1. Integer points in Hichin moduli space

Let Γ_g be the genus g surface group, consider $Hom(\Gamma_g, G)$. $Hit_{n,g} = \{$connected component of $\rho_n h$ where $h \in Hom(\Gamma_g, SL(2,\mathbb{R}))\}$. Here ρ_n is the n-dimensional irreducible representation $SL(2) \to SL(n)$. $Max_{n,g} = \{\pi : \Gamma_g \to Sp(2n,\mathbb{R}) | \int_{S_g} f_\pi^*(\omega) = n(29-2)\}$ where ω is the Kahler form.

Relation with geometric group theory: Theorem: every Hitchin representation is a quasi-isometric embedding. Same holds for maximal reps.

Theorem (Laboviz) MCG_g acts properly discontununously on both Hit and Max.

In higher Teichmuller theory, there is Presture metric analogous to Weil-Peterson metric, (but it is unknown if $MCG\backslash Hit$ and $MCG\backslash Max$ have finite volume) Bonahan-Drager coordinates analogous to Shearing coordinates, and a kind of Collaring lemma due to Zhang, Lee.

The notion of integer points: For $\pi \in Hit$ (same for Max), π is integral ($\in Hit(\mathbb{Z})$) if up to conjugation $im(\pi) \subset SL_n(\mathbb{Z})$. π is trace-integral ($\in Hit^{\mathbb{Z}}$) if its trace is in $\mathbb{Z}$. Furthermore, fix $\Lambda \subset SL(n,\mathbb{R})$, π is Λ-integral if $im(\pi) \subset \Lambda$ up to conjugate. The question is to count the MCG orbits in these sets.

$n = 2$: $Hit(\mathbb{Z}) = \emptyset$ and $MCG\backslash Hit^{\mathbb{Z}}$ is finite.

('95, Dechant) If Λ is word-hyperbolic, then for fixed g, there are only finitely many homomorphisms $\Gamma \to \Lambda$ up to conjugation.

(p.s. Kahn-Marboric gave a lower bound on the number of homomorphisms from Γ_g to $\pi_1(M^3)$.)

$n = 3$: $\exists$ family of representations from $\Delta(3,3,4) = \{a, b | a^3 = b^3 = (ab)^4 = e\}$ to $SL(3,\mathbb{R})$ parametrized by t that are not conjugate, and its image is in $SL(3,\mathbb{Z})$ when $t \in \mathbb{Z}$. This is done through a "bending" construction. Hence, $Hit_{3,g}(\mathbb{Z})$ is a infinite set, and same for $n \geq 5$ odd.

Definition of a height function (M-C-W) Let $Fuch_{n,g}$ be the Fuchsian locus, define $A(\pi,\rho) = \limsup_\gamma \frac{\log(\Lambda(\rho(\gamma)))}{\log(\Lambda(\rho(\rho)))}$, where Λ is the spectral radius, and $A(\rho) = \inf_{\pi \in Fuch} A(\pi,\rho)$.

Theorem: When A is bounded from above, $MCG\backslash Hit^{\mathbb{Z}}$ and $MCG\backslash Max^{\mathbb{Z}}$ are both finite.

However, $1/h(\rho)$ where h is the entropy won't work because the bending doesn't make entropy $\to 0$.

Another possible way of measuring height is to consider the minimal area of the immersion $S_g \to \Lambda\backslash SL(n,\mathbb{R})/SO(n)$ induced by the representation.