# 1. Random Dynamics and a formula for Furstenberg, Kullback-Ledrappier Entropy

Joint work with Brown.

Problem: understand the existence and properties of invariant measure under $\Gamma$ (a free group of 2 elements.)

Approach: stationary measure ($\mu = \mu * \gamma$). Show its invariance by showing that it is nice enough.

Homogenuous setting:

Homogenuous setting (BFLM, BQ): $\Gamma \subset SL(n, \mathbb{Z})$ large enough, generated by $supp(\gamma)$, acting on $\mathbb{T}^n$, then every stationary measure is Haar or atomic.

Application: Theorem 1 (BH) $\Gamma \subset SL(2, \mathbb{Z})$, assume $\mathbb{Z}$ is not a subgroup of finite index of $\Gamma$, $S = \{A_1, \dots, A_n\}$ a finite generating set, $f_i$ be $C^1$-small area-preserving perturbations to $A_i$, then a measure stationary under them is either the area form or atomic.

Theorem 2: $R_1, \dots R_m \in SO(3)$, with counting measure, $f_i$ are area-preserving perturbations to $R_i$ as elements in $Diff(S^2)$, then stationary implies area form or atomic.

Oseledet's theorem: $\gamma$ a finite supported measure on $Diff^r(M) = \Omega$, consider $F : \Omega^{\mathbb{N}} \times M \to \Omega^{\mathbb{N}} \times M$ as $(w, x) \mapsto (\sigma(w), f_0(x))$, where $\sigma$ is the shift and $f_0$ is the first entry of $w$. Then $\mu$ is stationary iff $F$ preserves $\gamma^{\mathbb{N}} \times \mu$. Similarly we can define $\hat{F}$ when replacing $\mathbb{N}$ with $\mathbb{Z}$.

Assume $\mu$ is ergodic, for $\gamma^{\mathbb{N}} \times \mu$ a.e. $(w, x)$, there is $0 \subset V_1 \cdots \subset V_k = E^S \cdots \subset T_x(M)$, s.t. $v \in V_i \backslash V_{i-1}$ then $\lim \frac{1}{n} \log |D(f_{n-1} \dots f_0)x| = \lambda_i$.

On $\gamma^{\mathbb{Z}} \times \mu$, there is decomposition $T_x M = \oplus_i E_i$. $V_i$ is tangent to foliations $\hat{W}_i$.

Furstenberg, Kullback-Ledrappier Entropy: measure the non-invariance of measure.

$f_* \mu = J_f \mu + \eta$, where $\eta \perp \mu$. $J_f = \frac{df_* \mu}{d\mu}$. Define $H_\gamma(\mu) = -\int \int \log J_f \times d\mu(x) d\gamma(f)$. $H = 0$ iff $\mu$ is invariant on $supp(\gamma)$.

Theorem: (Ledrappier, Avila-Viena) $H_\gamma(\mu) \leq \Lambda_- dim(M)$, where $\Lambda_- = \max_{\lambda_i < 0}(-\lambda_i)$.

Corr: if $\mu$ has no negative exponent then it is invariant.

Exercise: If all exponents are positive, the measure is atomic.

How to handle $H$ when there is negative exponent?

Let $\hat{\mu}_w$ be the conditional measure of $\hat{\mu} = \gamma^{\mathbb{Z}} \times \mu$ on $\{w\} \times M$, $\hat{\mu}^{\hat{W}_i}(w,x)$ be the conditional measure on $\hat{W}_i(w,x)$, $\mu^{\hat{W}_i}(w,x)$ be the conditional measure of $\mu$ on $\hat{W}_i$, the Hausdorff dimension of the latter 2 $\delta_i(\hat{\mu})$, $D_i(\mu)$, let $e_1 = \delta_1$, $e_i = \delta_i - \delta_{i-1}$ when $i > 1$, and define $E_i$ similarly from $D_i$, then:

$$H_\gamma = -\sum_{\lambda_i<0} (-\lambda_i)(E_i - e_i)$$

Remark: the sum depends only on the stable part.

Remark: $\hat{\mu}_w = \hat{\mu}_v$ if $w$ and $v$ have the same future. $\mu$ is the integral of these $\hat{\mu}_w$, hence is generally "thicker" that $\hat{\mu}_w$, hence the right-hand-side is usually positive.

Theorem: $\nu$ compactlu supported on $Diff^2(M)$, $dim(M) = 2$, assume $\mu$ has 1 positive & 1 negative exponent and ergodic, then either (1) $\mu$ is atomic (2) $\hat{\mu}^{W^u}$ is Lebesgue, (3) $E^s$ is non-random (i.e. $E^s(w,x)$ only depends on $x$ for a.e. $(w,x)$.