# Bottom of spectrum and equivariant families of boundary measures in negative curvature.

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Let $(\widetilde{M}, g)$ be an open, connected, complete Riemannian manifold, $\Delta = \mathrm{div}\nabla$ the Laplace Beltrami operator on $C^2$ functions.

Then [Sullivan (87)], there is $\lambda_0 \geq 0$ such that

1) The spectrum of $\Delta$ on $L^2(\widetilde{M}, \mathrm{Vol})$ is contained in $(-\infty, -\lambda_0]$, but not in $(-\infty, -\lambda_0 - \varepsilon]$, for $\varepsilon > 0$.

2) For any $\lambda \leq \lambda_0$, there are positive $(-\lambda)$-eigenfunctions.

From the spectral theorem, one gets expressions for $\lambda_0$

as the *Rayleigh quotient*:

$$\lambda_0 = \inf_{f \in C_c^2(\widetilde{M})} \frac{\int \|\nabla f\|^2}{\int f^2}.$$

as the decay rate of the *heat kernel*:
for $t > 0, x, y \in \widetilde{M}$, $p(t,x,y) := e^{t\Delta}(x,y)$ and

$$\lambda_0 = - \lim_{t \to \infty} \frac{1}{t} \ln p(t,x,y).$$

Example: $(\widetilde{M}, g) = (\mathbb{R}^d, g_0)$. Then, $\lambda_0 = 0$,

the spectrum of $\Delta$ is $(-\infty, 0]$ (by *Fourier transform*)

$$< \Delta f, g >_{L^2} = -\frac{1}{(2\pi)^d} \int |\xi|^2 \widehat{f}(\xi) \widehat{g}(\xi).$$

and for $\lambda \geq 0, \lambda = r^2$, and $\mu$ measure on $\mathbb{S}^{d-1}$,

$$F(x) := \int_{\mathbb{S}^{d-1}} e^{r<\xi, x>} d\mu(\xi)$$

is a $\lambda$-eigenfunction.

Example: $(\mathbb{H}^d, g_0)$ the hyperbolic space with curvature -1. Then, $\lambda_0 = (d-1)^2/4$, the spectrum is $(-\infty, -(d-1)^2/4]$ (use *representation theory* of $Iso_+(\mathbb{H}^d, g_0)$)

and for $\lambda \leq \frac{(d-1)^2}{4}, \lambda = -(d-1)^2 s(s-1), s \geq 1/2$, $\mu$ measure on $\mathbb{S}^{d-1}$ (seen as the *boundary* of $\mathbb{H}^d$) and $k(o,z,\xi)$ the *Poisson kernel*,

$$F_{s,\mu}(z) := \int_{\mathbb{S}^{d-1}} k^s(o,z,\xi)\,d\mu(\xi)$$

is a $(-\lambda)$-eigenfunction.

In both cases, when the measure $\mu$ is the rotation invariant measure $d\xi$, we get an eigenfunction that depends only on the distance to $o$.
In particular, the $(-\lambda_0)$-eigenfunction

$$F_{1/2,d.}(z) := \int_{\mathbb{S}^{d-1}} \sqrt{k(o,z,\xi)}\, d\xi$$

is called the ground state, or Harish-Chandra function.

In this talk, we assume that $(\widetilde{M}, g)$ is the universal cover of a compact manifold with negative sectional curvatures. Let $G$ be the group of deck transformations.

## GEOMETRY OF $\widetilde{M}$ AND $\partial\widetilde{M}$

$\widetilde{M}$ is homeomorphic to $\mathbb{R}^d$.

Geodesic rays $\gamma_1, \gamma_2$ are said to be equivalent if their Hausdorff distance is bounded.

The space $\partial\widetilde{M}$ of equivalence classes is homeomorphic to $\mathbb{S}^{d-1}$ and $\widetilde{M} \cup \partial\widetilde{M}$ is a closed $d$-dimensional ball.

For any $x \in \widetilde{M}$ and a suitable $a > 0$, $e^{-a d_x(\xi,\eta)}$, where $d_x$ is the Gromov product

$$d_x(\xi, \eta) := \lim_{y \to \xi, z \to \eta} \frac{1}{2} \left( d(x,y) + d(x,z) - d(y,z) \right)$$

defines a metric on $\partial\widetilde{M}$.

For all $x \in \widetilde{M}$,

$\pi_x : S_x\widetilde{M} \to \partial\widetilde{M}, \pi_x(v) := [\gamma_v(\mathbb{R}_+)] = \gamma_v(+\infty)$

defines a bi-Hölder continuous homeomorphism. We identify $\widetilde{M} \times \partial\widetilde{M}$ with $S\widetilde{M}$ by

$$(x, \xi) \sim \pi_x^{-1}(\xi).$$

The action of the covering group $G$ extends to $\partial\widetilde{M}$. The quotient $\widetilde{M} \times \partial\widetilde{M}/G$ identifies with $SM$.

The foliation $\widetilde{M} \times \{\xi\}, \xi \in \partial\widetilde{M}$ on $\widetilde{M} \times \partial\widetilde{M}$ descends to the *stable foliation* of the geodesic flow on $SM$.

*Regular equivariant families of boundary measures; r.e.m.*

A r.e.m. is a mapping $x \mapsto \mu_x, x \in \widetilde{M}, \mu_x$ measure on $\partial\widetilde{M}$ that is
1) equivariant: for all $g \in G, x, \mu_{gx} = g_*\mu_x$,
2) such that all $\mu_x$ have the same zero sets,
3) for fixed $x, y$, $\xi \mapsto \frac{d\mu_y}{d\mu_x}(\xi) =: k(x,y,\xi)$ is Hölder continous and
4) for fixed $\xi$, $y \mapsto k(x,y,\xi)$ is $C^1$.

We normalize a r.e.m. by $\int_{M_0} \mu_x(\partial\widetilde{M})dx = 1$, where $M_0$ is a fundamental domain for $G$.

Associated to a r.e.m. is a closed 1-form $\alpha(x,\xi)$. For each $\xi$, set

$$\alpha(x,\xi) := d \ln k(x_0, y, \xi)|_{y=x}.$$

It is well-defined by 4), is a closed 1-form on $\widetilde{M}$, does not depend on $x_0$ and is $G$-invariant:

$$g^*\alpha(gx, g\xi) = \alpha(x,\xi).$$

In particular, it descends on $\widetilde{M} \times \partial\widetilde{M} = SM$ as a closed 1-form along the stable foliation.

The *energy* $\mathcal{E}$ of a (normalized) r.e.m. is given by

$$\mathcal{E}(\mu) := \int_{M_0} \int_{\partial\widetilde{M}} \|\alpha(x,\xi)\|_x^2 \, d\mu_x(\xi) dx.$$

Examples:

The Margulis (or Patterson-Sullivan) r.e.m. $\nu_x$ is constructed as follows. For $R > 0$, let $\nu_{x,R}$ be the Lebesgue measure on the sphere of center $x$ and radius $R$.
Margulis (70) showed that there is a number $V$ such that the measures $e^{-VR}\nu_{x,R}$ weak* converge on $\widetilde{M} \cup \partial\widetilde{M}$ to a measure $\nu_x$ supported by $\partial\widetilde{M}$. Equivariance is clear.

$V$ is the volume entropy of $\widetilde{M}$ and the topological entropy of the geodesic flow; $\alpha$ is $V\times$ the Liouville 1-form on $SM$ and $\mathcal{E}(\nu) = V^2$.

The Lebesgue (or visibility) r.e.m. $\lambda_x$ is the image of the Lebesgue measure on $S_xM$ by $\pi_x$. Equivariance is clear. 2), 3) and 4) are due to Anosov (67).

The harmonic r.e.m. $\omega_x$ is the exit measure of the Brownian Motion starting from $x$. Equivariance and 2) are clear. 3) is due to Anderson and Schoen (85). 4) follows since $y \mapsto k(x, y, \xi)$ is harmonic.
The energy $\mathcal{E}(\omega)$ is $h$, the Kaimanovich entropy of the Brownian Motion.

All three families coincide (up to normalization) when $(\widetilde{M}, g)$ is a symmetric space. In the general variable negative curvature case, we have

**Theorem** [Mohsen 07]

$$4\lambda_0 = \inf\{\mathcal{E}(\mu); \mu \text{ r.e.m.}\}.$$

**Theorem 1** [L- Lim 15] *There exists a unique normalized r.e.m. $\mu^0$ such that*

$$\mathcal{E}(\mu^0) = 4\lambda_0.$$

Actually, Mohsen was considering the *Rayleigh quotient* of a normalized r.e.m. $\mathcal{R}(\mu)$

$$\mathcal{R}(\mu) := \int_{M_0} \int_{\partial \widetilde{M}} \|\nabla \sqrt{k(x_0, x, \xi)}\|_x^2 \, d\mu_{x_0}(\xi) dx.$$

We have:

$$\begin{aligned}
\mathcal{R}(\mu) &= \int_{M_0} \int_{\partial \widetilde{M}} \frac{\|\nabla \sqrt{k(x_0, x, \xi)}\|_x^2}{(\sqrt{k(x_0, x, \xi)})^2} \, d\mu_x(\xi) dx \\
&= \frac{1}{4} \int_{M_0} \int_{\partial \widetilde{M}} \|\nabla \ln k(x_0, x, \xi)\|_x^2 \, d\mu_x(\xi) dx \\
&= \frac{1}{4} \mathcal{E}(\mu).
\end{aligned}$$

We give one construction of $\mu_x^0, x \in \widetilde{M}$.

**Fact** [Sullivan 87] *The Green function of* $\Delta + \lambda_0$ *is finite:*

$$G_{\lambda_0}(x,y) = \frac{1}{\Delta + \lambda_0}(x,y) = \int_0^\infty e^{\lambda_0 t} p(t,x,y)dt,$$

where $p(t,x,y) = e^{t\Delta}(x,y)$ is the heat kernel on $\widetilde{M}$.

We prove:

**Fact** [L- Lim 15] *The measures* $G^2_{\lambda_0}(x,.)\nu_{x,R}$ *weak* converge on* $\widetilde{M} \cup \partial\widetilde{M}$ *to a measure* $\mu^0_x$ *supported by* $\partial\widetilde{M}$.

Then,

$$\frac{d\mu^0_y}{d\mu^0_x}(\xi) = \lim_{z \to \xi} \frac{G^2_{\lambda_0}(y,z)}{G^2_{\lambda_0}(x,z)} = k^2_{\lambda_0}(x,y,\xi),$$

where $k_{\lambda_0}(x,y,\xi)$ is a positive $(-\lambda_0)$-eigenfunction.

The last equality is

**Theorem 2** [L- Lim 15] *The Martin boundary of the operator* $\Delta + \lambda_0$ *is the geometric boundary* $\partial\widetilde{M}$.

Martin boundary for $\Delta + \lambda$:

Anderson-Schoen (85) for $\Delta$,

Ancona (85) for $\Delta + \lambda, \lambda < \lambda_0$ and for random walks with finite support on hyperbolic groups, the Green function being $G_r(g) := \sum_n r^n \mu^{*n}(g), r < R, R$ critical exponent.

Gouëzel (2014) for symmetric random walk on hyperbolic groups with finite support and $G_R$.

Our Theorem 2 is the Brownian motion counterpart of Gouëzel's result.

Theorem 2 shows that if indeed the measures $G^2_{\lambda_0}(x,.)\nu_{x,R}$ weak* converge to a measure $\mu^0_x$, then $\sqrt{\frac{d\mu^0_y}{d\mu^0_x}(\xi)}$ is a positive $(-\lambda_0)$-eigenfunction.

It follows that $\mathcal{E}(\mu^0) = 4\lambda_0$. Uniqueness follows by the strict convexity of the functional $\mu \mapsto \mathcal{E}(\mu)$.

In order to show weak* convergence of the measures $G^2_{\lambda_0}(x,.)\nu_{x,R}$, define for $v \in SM$, and $\widetilde{v}$ any lift o$v$ to $S\widetilde{M}$:

$$\varphi := -2 \lim_{t \to 0} \frac{1}{t} \ln k_{\lambda_0}(\gamma_{\widetilde{v}(0)}, \gamma_{\widetilde{v}}(t), \gamma_{\widetilde{v}}(+\infty)).$$

Then (Hamenstädt), $\varphi$ is a Hölder continuous function on SM. The Gibbs measure is mixing and replaces the measure of maximal entropy in Margulis's argument. There is no exponential factor since

**Fact** *Pressure* $(\varphi) = 0$.

The following positive $(-\lambda_0)$-eigenfunction $c(x,y)$ is the *ground state* around $x$:

$$\begin{aligned} c(x,y) &:= \int_{\partial\widetilde{M}} k_{\lambda_0}(x,y,\xi)\,d\mu_x^0(\xi) \\ &= \int_{\partial\widetilde{M}} \sqrt{d\mu_y^0}\sqrt{d\mu_x^0}. \end{aligned}$$

It appears in the Local Limit Theorem:

**Theorem 3** [L- Lim 15] *There is a constant* $C$ *such that*

$$\lim_{t\to\infty} t^{3/2} e^{\lambda_0 t} p(t,x,y) = \frac{C}{\sqrt{\pi}}\, c(x,y).$$

LLT for symmetric spaces is due to P. Bougerol (81).

Analogous LLT for random walks on a countable group $G$ with probability $\mu$:

P. Gerl (79): $G = \mathbb{F}_2$, $\mu$ on generators;

P. Gerl & W. Woess (86): $G = \mathbb{F}_d$, $\mu$ on generators;

S. Lalley (93): $G = \mathbb{F}_d$, $\mu$ finite support;

S. Gouëzel & S. Lalley (13): $G$ surface group, $\mu$ symmetric finite support;

S. Gouëzel (14)(15): $G$ fin.gen. Gromov-hyperbolic group, $\mu$ symmetric superexponential moments.

Local Limit Theorem follows from a *Tauberian Theorem* and

$$\lim_{\lambda \nearrow \lambda_0} \sqrt{\lambda_0 - \lambda}\, \frac{\partial}{\partial \lambda} G_\lambda(x,y) = C\, c(x,y). \qquad (1)$$

The proof of (1) uses the Hölder continuous function

$$\varphi_\lambda := -2 \lim_{t \to 0} \frac{1}{t} \ln k_\lambda(\gamma_{\widetilde{v}(0)}, \gamma_{\widetilde{v}}(t), \gamma_{\widetilde{v}}(+\infty)).$$

Then, $\varphi_\lambda \to \varphi$ as $\lambda \to \lambda_0$, the Pressure $P_\lambda$ of $\varphi_\lambda$ goes to 0 as $\lambda \to \lambda_0$ and

**Fact** *The Gibbs states $m_\lambda$ of $\varphi_\lambda$ are uniformly mixing on Hölder continuous functions as $\lambda \to \lambda_0$.*

$$-P_\lambda \frac{\partial}{\partial \lambda} G_\lambda(x,y) = -P_\lambda \int_{\widetilde{M}} G_\lambda(x,z) G_\lambda(z,y) dz =$$

$$-P_\lambda \int_0^\infty e^{P_\lambda R} \left( \int_{S_R(x)} e^{-P_\lambda R} \frac{G_\lambda(y,z)}{G_\lambda(x,z)} G_\lambda^2(x,z) dz \right) dR,$$

where $S_R(x)$ is the sphere of radius $R$ and $dz$ the corresponding Lebesgue measures.

As $R \to \infty$, $\frac{G_\lambda(y,z)}{G_\lambda(x,z)}$ converges to $k_\lambda(x, y, \pi_x(v_x^z))$ (Martin Boundary),

$$a_{\lambda,R}(x,y) := \int_{S_R(x)} e^{-P_\lambda R} k_\lambda(x,y,\pi_x(v_x^z)) G_\lambda^2(x,z)\,dz$$

converges, as $R \to \infty$ to $\int_{\partial \widetilde{M}} k_\lambda(x,y,\xi)\,d\mu_x^\lambda(\xi)$ for some r.e.m. $\mu_x^\lambda$ (by mixing of the Gibbs measure $m_\lambda$).

Using the uniform mixing of $m_\lambda$, as $\lambda \to \lambda_0$,

$$-P_\lambda \frac{\partial}{\partial \lambda} G_\lambda(x,y) = -P_\lambda \int_0^\infty e^{P_\lambda R} a_{\lambda,R}(x,y)\,dR$$

converges towards

$$\lim_{\lambda \to \lambda_0, R \to \infty} a_{\lambda,R}(x,y) = \int_{\partial \widetilde{M}} k_{\lambda_0}(x,y,\xi)\,d\mu_x^{\lambda_0}(\xi).$$

The r.e.m. $\mu^0$ is the normalized $\mu^{\lambda_0}$ so that

$$\int_{\partial\widetilde{M}} k_{\lambda_0}(x,y,\xi)d\mu_x^{\lambda_0}(\xi) = c(x,y)\int_{M_0} \mu_z^{\lambda_0}(\partial\widetilde{M})dz.$$

In order to eliminate $P_\lambda$, one uses in the same way

$$\begin{aligned} & (-P_\lambda)^3 \frac{\partial^2}{\partial\lambda^2} G_\lambda(x,y) \\ = \;& 2(-P_\lambda)^3 \int_{\widetilde{M}} G_\lambda(x,z)G_\lambda(z,w)G_\lambda(z,y)\,dzdw \end{aligned}$$

and the *uniform 2-mixing* of $m_\lambda$ as $\lambda \to \lambda_0$.

Finally, one gets, setting $F(\lambda) = \frac{\partial}{\partial\lambda} G_\lambda(x,y)$, as $\lambda \to \lambda_0$,

$$\frac{2F'(\lambda)}{F(\lambda)^3} \to (Cc(x,y))^{-2},$$

which shows (1).

The constant $C$ is related to the *total mass* of the r.e.m. $\mu^0$ as follows.

**Fact** *For $x \in \widetilde{M}, \xi \neq \eta \in \partial\widetilde{M}$, the following limit exists:*

$$\theta_x(\xi,\eta) := \lim_{y\to\xi, z\to\eta} \frac{G_{\lambda_0}(y,z)}{G_{\lambda_0}(y,x)G_{\lambda_0}(x,z)}.$$

**Fact** *The measure* $m^0$ *on* $(\partial\widetilde{M})^3$ *defined by* $dm^0(\xi,\eta,\zeta) :=$

$$\theta_x(\xi,\eta)\theta_x(\eta,\zeta)\theta_x(\eta,\xi)d\mu_x^0(\xi)d\mu_x^0(\eta)d\mu_x^0(\zeta)$$

*does not depend on* $x$. *In particular,* $m^0$ *is* $G$*-invariant.*

Let $\Upsilon$ be the $m^0$-measure of a fundamental domain for the action of $G$ on $(\partial\widetilde{M})^3$. Then,

$$C = \frac{1}{2\sqrt{\Upsilon}}.$$

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