MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Chenxi Wu **Email/Phone:** cw538@cornell.edu

Speaker's Name: Jean-François Quint

Talk Title: Random walks on semisimple groups

Date: May / 15 / 2015 **Time:** 9 : 00 **am** / pm (circle one)

List 6-12 key words for the talk: random walk on groups, stationary measure, polynomial moment unimodular, growth

Please summarize the lecture in 5 or fewer sentences: Overview on the random walk on semisimple group, especially when there isn't exponential moment.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

1. RANDOM WALKS ON SEMISIMPLE GROUPS

μ a probability measure on $GL(V)$, V a finite dimensional vector space. WE call μ has exponential moment if $\int \max(||g||, ||g^{-1}||)^\delta d\mu < \infty$ for some $\delta > 0$, has polynomial moment if $\int \log(\max(||g||, ||g^{-1}||))^\delta d\mu < \infty$ for some $\delta > 0$.

Let Γ_μ be the closed subgroup spanned by the support of μ. Γ_μ is irreducible if all Γ-invariant subspaces of V are 0 and V. It is totally irreducible if there are no trivial collection of subspaces invariant under Γ. If the action is not totally irreducible, it is always possible to pass to a subgroup of finite index & a subspace.

Also assume that Γ is proximal (i.e. it contains a proximal element, i.e. an element of the form $diag(a, h)$ where $a \in \mathbb{R}^*$ and the spectral radius of h is smaller than $|a|$). If G is not proximal, pass to the r-th exterior product of V.

Theorem (Fursternberg) under these assumptions. there is a unique μ-stationary measure ν on $\mathbb{P}V$.

Let $(B, \beta) = (G, \mu)^{\otimes \mathbb{N}}$, T is the shift map $B \to B$, ν is μ stationary means the map $B \times \mathbb{P}V \to B \times \mathbb{P}V$, $(b, x) \mapsto (Tb, b_1 x)$ preserves $\beta \otimes \nu$. We need to use ergodic theorem on $(b, x) \mapsto \log \frac{||b_1 v||}{||v||}$.

Theorem (B-Q): If μ has polynomial moment of order p, ν has positive Hausdorff dimension. Furthermore, $\forall y \in \mathbb{P}V^*$, $\delta(x, y)$ is the distance from x to $y^\perp$, then $\int_{\mathbb{P}V} |\log \delta(x, y)|^{p-1} d\nu(x) \leq C$.

In dimension 2 case $p-1$ can be replaced with p. It is unknown for the other case. From this one can prove central limit theorem and the law of large numbers.

Proposition: μ is a measure on $GL(V)$, $\Gamma_\mu = < supp \mu >$ and H_μ is its Zariski closure. Assume that H_μ is connected and semisimple, $L < H_\mu$ algebraic and unimodular, then $\exists t > 0$, for any $x \in H_\mu / L$, $\forall K \subset H_\mu / L$ compact, $Prob(g_n \dots g_1 x \in K) \to 0$.

Lemma: μ is a measure on $GL(V)$ with polynomial moment, $\Gamma_\mu = < supp \mu >$ and H_μ is its Zariski closure. Assume that H_μ is connected and semisimple, $L < H_\mu$ algebraic and unimodular, then $\exists t > 0$, for Lebesgue a.e. $x \in H_\mu / L$, $\forall K \subset H_\mu / L$ compact, $Prob(g_n \dots g_1 x \in K) \sim e^{-tn}$.

The proof is due to the fact that unimodular implies $\phi \mapsto \mu * \phi : L^2(H_\mu / L) \to L^2(H_\mu / L)$ has spectral radius smaller than 1.

Remark: This is not true without "a.e". When $H = SL(2,\mathbb{R})$, $L = \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\}$, it is possible to choose a μ that decays very slowly but is still of polynomial moment that makes $Prob(g_n \dots g_1 x \in K)$ decreases subexponentially for an x.

Proposition: μ is a measure on $G = SL(2,\mathbb{R})$ with polynomial moment, let A be the diagonal group, $\forall x \in G/A$, $\forall K \subset G/A$ compact, $Prob(g_n \dots g_1 x \in K) \sim e^{-tn}$ for some t.

Question: Is polynomial moment necessary? How about $SL(2,\mathbb{C})/SL(2,\mathbb{R})$?