1. Dynamics of Moduli space of flat structure

Consider $SL(2, \mathbb{R})$ action on $H(a_1, \dots a_k) = \{(X, \omega)\}$, where X is a Riemann surface and ω is a holomorphic 1-form hence defines a flat structure with mild singularities. Geodesics on it that don't pass through the singularities form cylinders. Existence of cylinders is proved by Masur and Vorabets through $SL(2, \mathbb{R})$ action, and their areas, lengths and other properties have been studied by Eskin, Mazur, Vorabets, Zorich etc.

The holonomy of ω on a triangulation of X gives a local coordinate of H.

The diagonal and unipotent subgroups of $SL(2, \mathbb{R})$ give the geodesic & horocycle flow. For geodesic flow, closed orbits are defined over fields of degree as most $2g$. Veech and Mazur defined an invariant measure on H, Forni-Veech showed the non-uniform hyperbolicity, and Thurston studied its stable and unstable submanifolds. The unipotent flow u_t is ergodic when area is normalized, is non-divergent due to Minsky-Weiss and its minimal set is calculated by Smillie-Weiss.

$SL(2, \mathbb{R})$ orbit closure: Theorem (EMM) $SL(2, \mathbb{R})$ orbit closures & invariant measures are affine PL submanifolds.

When $g = 2$ they are constructed by Calta and McMullen and classified by McMullen.

Problems: 1. Horocycle orbit closure and invariant measure: there are examples found by Smillie-Weiss.

2. Rates of equidistribution in $SL(2, \mathbb{R})$ orbit closures.

3. Finiteness results: $g = 3$, algebraically primitive case, and $H^{hyp}(g-1, g-1)$ were studied by Bambridge, Moller and Habbeger, rank-1 case was studied by Wright, Lanneau, Nguyen, Fillip, and prime genus case was studied by Matheus and Wright. Wright also found relationship between general orbit closures and closed orbits in them.

There are two important invariants that characterizes the orbit closure M: (1) rank, which is 1/2 of the dimension of TM under the projection to absolute cohomology, and (2) field of definition.

Theorem (Wright, cylinder deformation theorem) C_i are parallel cylinders on $q \in M$, then M is invariant under their deformation.

Theorem (M-Wright) The boundary of orbit closure M is of the "expected dimension".

Theorem: M is affine, then $R(M) =$ the set of ratios of the length or moduli of 2 parallel generic cylinders is finite.

Theorem (M-Wright) If $rank(M) = g$, $M = H$ or the hyperelliptic locus.

Remark: this should be true for $rank(M) > g/2 + 1$.

As a consequence, if M has no defining relation that does not involve relative cohomology, M is H or the hyperelliptic locus.

The non-divergence of horocycle flow plays a crucial rule in the proofs of these results.