Introduction to Bounded Cohomology and Applications to Rigidity

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(based on a talk by Alessandra Iozzi)

1 Introduction

1.1 Example of the First Use of Bounded Cohomology

- The first use of bounded cohomology is due to Gromov [Gromov 80], as a tool to control the minimal volume of M

$$\min \text{vol}(M) := \min\{\text{vol}(M) : -1 \leq \kappa \leq 1\}$$

in terms of the simplicial volume $||M||$ of M, that is the ℓ^1-norm of the fundamental class $[M]$

$$\|M\| := \big\|[M]\big\|_1 = \inf\{|c|_1 : c \in C_n(M, \mathbb{R}) \text{ fundamental cycle}\}.$$

- Ghys [Ghys 87] gave a classification of actions of a finitely generated group Γ by homeomorphisms on a circle, via the *bounded Euler class.* This was one of the first applications using bounded Euler class.

- Bavard [Bavard 91] gave a characterization of finitely generated group Γ with vanishing stable commutator length, via the comparison between bounded and ordinary cohomology.

- Mineyev [Mineyev 00] gave a characterization of Gromov hyperbolic groups, via the comparison between bounded and ordinary cohomology.

1.2 Bounded Cohomology via an Homological Algebra approach

- Ivanov [Ivanov 87] Used homological algebra with finitely generated groups and trivial coefficients.

- Noskov [Noskov 90] Used homological algebra with finitely generated groups with coefficients.

- Burger and Monod [Burger-Monod 00] were able to fruitfully use bounded cohomology to look at locally compact groups with coefficients.

2 Definitions and Properties

2.1 Definitions

DEFINITION 2.1. Given a locally compact group G, we define the space of real-valued continuous bounded functions on the cartesian product G^n

$$C_b(G^n, \mathbb{R}) := \{f : G^n \to \mathbb{R} : \text{ continuous and } \|f\|_\infty < \infty\} \tag{1}$$

with the diagonal G-action

$$(hf)(g_0, ..., g_n) := f(h^{-1}g_0, ..., h^{-1}g_n)$$

that makes it into a G-module. If $C_b(G^n, \mathbb{R})^G$ is the submodule of G-invariant vectors, we define the *homogeneus coboundary operator*

$$d_n : C_b(G^{n+1}, \mathbb{R})^G \to C_b(G^{n+2}, \mathbb{R})^G$$

by

$$(d_n f)(g_0, ..., g_{n+1}) := \textstyle\sum_{j=0}^{n} (-1)^j f(g_0, ..., \hat{g}_j, ..., g_{n+1}).$$

It is easy to check that $d_{n+1}d_n = 0$, so that $\operatorname{Im} d_n \subseteq \ker d_{n+1}$.

DEFINITION 2.2. Bounded cohomology is the cohomology of the complex

$$0 \to C_b(G, \mathbb{R})^G \xrightarrow{d_0} C_b(G^2, \mathbb{R})^G \xrightarrow{d_1} C_b(G^3, \mathbb{R})^G \xrightarrow{d_2} ..., \tag{2}$$

that is,

$$H^n_{cb}(G, \mathbb{R}) := \frac{ZC_b(G^{n+1}, \mathbb{R})^G}{BC_b(G^{n+1}, \mathbb{R})^G}$$

where $ZC_b(G^{n+1}, \mathbb{R})$ are the cocycles,

$$ZC_b(G^{n+1}, \mathbb{R})^G := \ker\{d_n : C_b(G^{n+1}, \mathbb{R})^G \to C_b(G^{n+2}, \mathbb{R})^G\}$$

and $BC_b(G^{n+1}, \mathbb{R})$ are the coboundaries,

$$BC_b(G^{n+1}, \mathbb{R})^G := \operatorname{im}\{d_{n-1} : C_b(G^n, \mathbb{R})^G \to C_b(G^{n+1}, \mathbb{R})^G\}.$$

REMARKS.

- In (**??**) it is necessary to take invariants otherwise the cohomology of the complex would be identically zero for all groups G and all n.

- One could replace $\mathbb{R}$ by a coefficient G-module E^*, where E^* is the dual of a separable Banach space E on which G acts by linear isometries, $G \to \mathrm{Isom}(E)$.

- If we do not require in (**??**) that the functions are bounded, we obtain the ordinary group cohomology. In this case the coefficients are topological vector spaces with a continuous G-action.

- If G is discrete then in (**??**) there are no continuity requirement and the bounded cohomology is denoted only by $H_b^n(G, \mathbb{R})$.

- What we gave above is the *homogeneous definition* of the continuous bounded cohomology. We could also give the *non-homogeneous* definition, by giving the *non-homogeneous coboundary operator*

$$\delta_n : C_b(G^n, \mathbb{R}) \to C_b(G^{n+1}, \mathbb{R})$$

defined as

$$(\delta_n f)(g_1, ..., g_{n+1}) := \\ f(g_2, ..., g_{n+1}) + \sum_{j=1}^n (-1)^j f(g_1, ..., g_j g_{j+1}, ..., g_{n+1}) + (-1)^{n+1} f(g_1, ..., g_n)$$

EXERCISE. Find the continuous maps

$$C_b(G^n, \mathbb{R})^G \overset{\rho^{\mathrm{n}-1}}{\underset{\tau^{\mathrm{n}}}{\rightleftarrows}} C_b(G^{n-1}, \mathbb{R})$$

such that the diagram

$$\begin{array}{ccc} C_b(G^n, \mathbb{R})^G & \xrightarrow{d_{n-1}} & C_b(G^{n+1}, \mathbb{R})^G \\ \uparrow \tau^n & & \downarrow \rho^n \\ C_b(G^{n-1}, \mathbb{R})^G & \xrightarrow{d_{n-1}} & C_b(G^n, \mathbb{R}) \end{array}$$

commutes.

2.2 Properties

We have the following essential features of continuous bounded cohomology:

1. There is a seminorm in bounded cohomology, that is if $\kappa \in H^n_{cb}(G,\mathbb{R}) \Rightarrow$ is a bounded cohomology class, then

$$\|\kappa\| := \inf\{\|c\|_\infty : [c] = \kappa\}.$$

   Note that in general $H^n_{cb}(G,\mathbb{R})$ is only a seminormed space.

2. (Gromov, Brooks) If M a CW-complex, then there is an isometric isomorphism

$$H^n_b(M,\mathbb{R}) \cong H^n_b(\pi_1(M),\mathbb{R}).$$

   This ia also true in ordinary cohomology, but only if the universal cover is contractible.

3. If G is *amenable* (e.g. compact, abelian, minimal parabolic, solvable, compact extension of solvable, etc.), then $H^n_{cb}(G,\mathbb{R}) = 0$ (Trauber)

We note here some differences between bounded and ordinary cohomology:

1. If $n = 0$, then bounded continuous and ordinary cohomology coincide, as $H^0_{cb}(G,\mathbb{R}) = \mathbb{R} = H^0_c(G,\mathbb{R})$.

2. If $n = 1$, then $\delta_1 f(g,h) = f(h) - f(gh) + f(g)$ and $BC_b(G,\mathbb{R}) = 0$, thus

$$H^1_{cb}(G,\mathbb{R}) = \{f : G \to \mathbb{R} : \delta_1 f = 0, f \text{ cont. and bounded}\} = \mathrm{Hom}_{cb}(G,\mathbb{R}) = 0.$$

   while $H^1_c(G,\mathbb{R}) = \mathrm{Hom}_c(G,\mathbb{R})$.

3. If $n = 2$, then continuous bounded cohomolgoy is already difficult to compute. If G is a non-compact simple Lie group with finite center, then

$$H^2_{cb}(G,\mathbb{R}) \cong H^2_c(G,\mathbb{R}) \cong \Omega^2(X)^G$$

   where X is the symmetric space associated to G (for example $G = PU(1,1)$ and $X = \mathbb{D}^2$ is the Poincaré disk).

   In general however $H^2_{cb}(G,\mathbb{R}) \neq H^2_c(G,\mathbb{R})$. For example,

$$H^2(\mathbb{F}_2, \mathbb{R}) = 0 \text{ but } \dim H^2_b(\mathbb{F}_2, \mathbb{R}) = \infty.$$

To see that $H^2(\mathbb{F}_2, \mathbb{R}) = 0$, recall that $H^2(G, \mathbb{R})$ is in a 1-1 correspondence with central extensions of G and all these split if $G = \mathbb{F}_2$. Alternatively, we also see this by looking at the bouquet M of two circles and using that $H^2(\mathbb{F}_2, \mathbb{R}) \cong H^2(M, \mathbb{R})$ and Mayer-Vietoris.

The fact that $\dim H^2_b(\mathbb{F}_2, \mathbb{R}) = \infty$ was shown by Brooks and Mitsumatsu with a complicated and non-transparent proof. Recently, however, Rolli [Rolli 09] showed that for every generator there is an isometric embedding $\ell^\infty_{odd}(\mathbb{Z}, \mathbb{R}) \hookrightarrow H^2_b(\mathbb{F}_2, \mathbb{R})$. In fact, let $\mathbb{F}_2 =< a, b >$. Choose $s_a, s_b \in \ell^\infty_{odd}(\mathbb{Z}, \mathbb{R})$, that is $s_a(-n) = -s_a(n)$ and $s_b(-n) = -s_b(n)$, and define $f : \mathbb{F}_2 \to \mathbb{R}$ by

$$f(a^{k_1}b^{h_1}...a^{k_n}b^{h_n}) := s_a(k_1) + s_b(h_1) + ... + s_a(k_n) + s_b(h_n).$$

Here it is easy to see that f is not bounded but $\delta_1 f$ is a bounded cocycle.

Summarizing, we saw that in degree two a lot of information can be obtained from the *comparison map*,

$$H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R}).$$

3 Examples

- Let $G = \text{Homeo}_+(S^1)$ (thought of as a discrete group, as it is not locally compact).

  If $x \in S^1$ is a fixed basepoint, $g_0, g_1, g_2 \in \text{Homeo}_+(S^1)$, we define the *orientation cocycle* by

  $$c(g_0, g_1, g_2) := \begin{cases} 1 & \text{if } (g_0x, g_1x, g_2x) \text{ is positively oriented} \\ -1 & \text{if } (g_0x, g_1x, g_2x) \text{ is negatively oriented} \\ 0 & \text{otherwise} \end{cases}$$ The cocycle c is obviously $\text{Homeo}_+(S^1)$-invariant and the bounded cohomology class it defines is a multiple of the *bounded Euler class* that will be defined in Kathryn Mann's talk..

- Let now $G = PU(1, 1)$, $\mathbb{D}^2$= unit disk with the Poincaré metric $(1-|z|^2)^{-2}|dz|^2$ and area form $\omega_{\mathbb{D}^2} = (1 - |z|^2)^{-2} dz \wedge d\overline{z}$.

  If $x \in \mathbb{D}^2$, we define

$$b_{\mathbb{D}^2}(g_0, g_1, g_2) := \int_{\triangle(g_0x, g_1x, g_2x)} \omega_{\mathbb{D}^2},$$

where $\triangle(g_0x, g_1x, g_2x)$ is the geodesic triangle with vertices g_0x, g_1x, g_2x. Then $b_{\mathbb{D}^2}(g_0, g_1, g_2)$ is a bounded cocycle, since $|b_{\mathbb{D}^2}(g_0, g_1, g_2)| < \pi$ and it is G-invariant.

If we choose instead $x \in \delta\mathbb{D}^2 \cong S^1$, we define $\beta_{\mathbb{D}^2}$ analogously by integration on ideal triangles. Then $\beta_{\mathbb{D}^2}$ takes only $\pm\pi$ and 0 as values and hence

$$\pi c|_{PU(1,1)} = \beta_{\mathbb{D}^2}.$$

- Analogously, we can consider $G = PO(1, n)^o$ and the cocycle defined by considering the volume of simplices in real hyperbolic n-space $\mathcal{H}^n_{\mathbb{R}}$. Then, since this volume is uniformly bounded and $G = PO(1, n)^o$-invariant, this defines a G-invariant alternating continuous bounded cocycle, this time in degree n. Likewise, the volume of *ideal* simplices in $H^n_{\mathbb{R}}$ (that is simplices with vertices on the sphere at infinity $\partial H^n_{\mathbb{R}}$ of $H^n_{\mathbb{R}}$) gives a G-invariant alternating bounded cocycle in degree n.

4 Homological Algebra Approach

4.1 A Different Resolution

Many of the cocycles defined above (in fact, all those on the boundary) are not continuous. To give flexibility to bounded cohomology, it is convenient to use cocycles that are not necessarily continuous and that do not live on the group itself. To this extent the following result is paramount:

Theorem 4.1 (Burger-Monod, 00).
There is an *isometric isomorphism*

$$H^n_{cb}(G, \mathbb{R}) \cong \frac{ZL^\infty_{alt}(B^{n+1},\mathbb{R})^G}{BL^\infty_{alt}(B^{n+1},\mathbb{R})^G},$$

where (B, ν) is a standard measure G-space with a quasi-invariant measure ν on which G acts amenably and $L^\infty_{alt}(B^{n+1}, \mathbb{R})^G$ consists of the G-invariant L^∞ (equivalence classes of) functions on B^{n+1}, which are *alternating*, that is such that

$$f(\sigma(b_0, \dots, n_n)) = \mathrm{sign}(\sigma) f(b_0, \dots, b_n).$$

Examples of Amenable Spaces.

- If G is a simple Lie group and $P < G$ is a minimal parabolic subgroup then the action of G on $(G/P, \nu)$ is amenable, where ν is the quotient of the Haar measure. For example one can take $G = SL(n, \mathbb{R})$ and P the subgroup of upper triangular matrices, so that G/P is the space of full flags.

- If T_r is the tree associated to $\mathbb{F}_r$, then $\mathbb{F}_r$ acts amenably on ∂T_r its boundary with respect to

$$\nu(C(x)) = \tfrac{1}{(2r(2r-1)^{n-1})}$$

  where $|x| = n$ and $C(x) \subset \partial T_r$ is the cone consisting of infinite words starting with x.

- Let Γ be a finitely generated group, $\rho : \mathbb{F}_r \to \Gamma$ a presentation with $N = \ker\rho$ and consider the subalgebra of N-invariant L^∞ functions on ∂T_r

$$L^\infty(\partial T_r)^N \subset L^\infty(\partial T_r).$$

  Mackey point realisation theorem asserts that there exists a standard measure space $(B, \bar{\nu})$ with a Γ-map

$$p : \partial T_r \to B,$$

  so that $p_*(\nu) = \bar{\nu}$ and there is an identification

$$L^\infty(\partial T_r)^N \xrightarrow{\cong} L^\infty(B)$$

  via p^*. Moreover $G \curvearrowright B$ amenably.

4.2 Amenability and Double Ergodicity

In all of the above cases the action is *doubly ergodic*, i.e. ergodic on $B \times B$. We recall that this is equivalent to saying that

$$\nexists \textbf{ non-constant invariant measurable functions } B \times B \to \mathbb{R}.$$

The isometric isomorphism in Theorem 4.1 in degree two reads

$$H^2_{cb}(G, \mathbb{R}) \cong \frac{ZL^\infty_{alt}(B^3, \mathbb{R})^G}{BL^\infty_{alt}(B^3, \mathbb{R})^G} = \frac{\ker\{d_2 : L^\infty_{alt}(B^3) \to L^\infty_{alt}(B^4)\}}{im\{d_1 : L^\infty_{alt}(B^2) \to L^\infty_{alt}(B^3)\}}.$$

By double ergodicity $L^{\infty}(B^2, \mathbb{R}) = \mathbb{R}$, so that $L^{\infty}_{alt}(B^2, \mathbb{R}) = 0$. It follows then that

$$H^2_{cb}(G, \mathbb{R}) \cong ZL^{\infty}_{alt}(B^3, \mathbb{R})^G$$

The upshot of the above results is that continuous bounded cohomology in degree 2 is a Banach space.

5 Applications

5.1 Computation of norms

By the above result, in degree two the seminorm is actually a norm and its calculation is very easy once one know the norm of the cocycle representing the class. For example, if we define the *bounded Kähler class* to be $\kappa^b_{\mathbb{D}^2} := [\beta_{\mathbb{D}^2}]$, then,

$$\kappa^b_{\mathbb{D}^2} = \inf\{||c||_\infty : [c] = \kappa^b_{\mathbb{D}^2}\} = ||\beta_{\mathbb{D}^2}||_\infty = \pi.$$

5.2 Milnor–Wood inequality and Teichmüller space

We now investigate the study of homomorphisms of lattices into Lie groups. The celebrated Margulis' superrigidity theorem [Margulis, '74] classifies all asserts that if $\Gamma < H$ is a lattice, $\mathrm{rk}(H) \geq 2$ (e.g. $H = SL(n, \mathbb{R})$, for $n \geq 3$), G is a simple non-compact Lie group and $\rho : \Gamma \to G$ is a homomorphism with $\overline{\rho(\Gamma)}^Z = G \Rightarrow \rho$, then ρ the restriction of a rational homomorphism of H, $\rho : H \to G$. The same holds if $\mathrm{rk}(H) = 1$ and if H has property (T) (that is, $H \neq SO(n, 1), SU(n, 1)$) [Corlette, '90]. In fact, if $H = SO(n, 1)$, not only Margulis' and Corlette's theorems do not apply, but actually superrigidity does not hold.

So we consider a different approach to study representations of lattices in $H = SO(2, 1) \cong SU(1, 1) \cong SL(2, \mathbb{R})$.

Let Σ_g be a compact orientable surface, and let $\Gamma_g := \pi_1(\Sigma_g)$. If $\kappa^b_{\mathbb{D}^2} \in H^2_{cb}(PU(1, 1), \mathbb{R})$, then $\rho^*(\kappa^b_{\mathbb{D}^2}) \in H^2_b(\Gamma_g, \mathbb{R}) \cong H^2_b(\Sigma_g, \mathbb{R})$, where we used the isomorphism discussed in § 2.2(2). We can use now the duality

$$< \cdot, \cdot >: H^2_b(\Sigma_g, \mathbb{R}) \times H^{\ell^1}_2(\Sigma_g, \mathbb{R}) \to \mathbb{R}$$

and obtain an invariant on the space of representations from Γ into H, called the *Euler number*

$$E(\rho) := | < \rho^*(\kappa^b_{\mathbb{D}^2}), [\Sigma_g] > |.$$

Using the computation of the norm above and of the simplicial volume of Σ_g, we obtain the *Milnor-Wood Inequality*

$$|E(\rho)| = |< \rho^*(\kappa^b_{\mathbb{D}^2}), [\Sigma_g] >| \leq \left\|\rho^*(\kappa^b_{\mathbb{D}^2})\right\| \left\|\Sigma_g\right\| \leq \left\|\kappa^b_{\mathbb{D}^2}\right\| \left\|\Sigma_g\right\| = 2\pi|\chi(\Sigma_g)|.$$

DEFINITION 5.1. A representation $\rho : \Gamma_g \to PU(1,1)$ is *maximal* if

$$E(\rho) = 2\pi|_\chi(\Sigma_g)|.$$

Perhaps using one of the many different definitions of the Euler number, it is easy to see that hyperbolizations (that is representations $\rho : \pi_1(\Sigma_g) \to SL(2,\mathbb{R})$ that are discrete an injective). The converse is true and not obvious, although by now classical.

THEOREM 5.2 (GOLDMAN, '80). ρ is *maximal* if and only if ρ is a *hyperbolization.*

5.3 Applications, Cont.

REMARKS.

- Analogous results for Σ with non-empty boundary
- *Teichmüller space* can be identified with the space of hyperbolizations, and hence the space of maximal representations. This leads to defining *higher Teichmüller theory* the generalisation of the above theory obtained by replacing $PU(1,1)$ by other Lie groups G. More specifically:
  - G split simple - e.g. $SL(n,\mathbb{R})$, $Sp(2n)$ (Hitchin, Labourie, Fock-Goncharov)
  - G Hermitian - e.g. $SU(p,q)$, $Sp(2n)$ (Burger-I.-Wienhard, Bradlow-Garcia Prada-Gothen)