Groups Acting on the Circle

Rigidity, Flexibility, and Moduli Spaces of Actions

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1. Introduction

In this talk, Γ is a finitely generated group, $\text{Homeo}_+(S^1)$ is the group of orientation preserving homeomorphisms of S^1, and $\text{Hom}(\Gamma, \text{Homeo}_+(S^1))$ is the space of representations of $\Gamma \to \text{Homeo}_+(S^1)$. This space has an important interpretation: if Γ is the fundamental group of a manifold M, then $\text{Hom}(\Gamma, \text{Homeo}_+(S^1))$ is the space of *flat S^1 bundles over M*.

Definition 1.1. A bundle is *flat* if it admits a flat connection, or equivalently if it admits a foliation transverse to the fibers.

The correspondence between $\text{Hom}(\Gamma, \text{Homeo}_+(S^1))$ and flat bundles is through the *monodromy representation*.

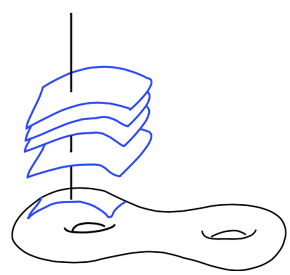

Figure 1: Unravelling the circle on top of a point on the torus and looking at transverse foliations

An important interpretation of these objects is that there is a monodromy representation,

$$\text{flat bundle} \longleftrightarrow \rho : \Gamma : \pi_1(M) \to \text{Homeo}_+(S^1)$$

and therefore

$$\text{flat bundles/equivalence} \longleftrightarrow \text{Hom}(\Gamma, \text{Homeo}_+(S^1))/\text{(semi)-conjugacy}$$

The following picture illustrates how to define the monodromy representation given a flat bundle: for each point p in S^1 (which we identify with the fiber over the basepoint in Σ), and each loop γ in $\pi_1(\Sigma)$, there is a unique lift of γ starting at p and tangent to the leaves of the foliation. We define $\rho(\gamma)(p)$ to be the other endpoint of this lifted path.

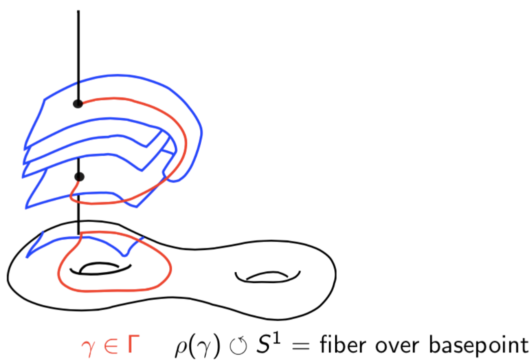

Figure 2: Lifting a loop γ to define the monodromy

2. The Basic Problem

How do we understand the space $\mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1))/\sim$? (from now on, $\sim$ means semi-conjugacy, but we won't worry too much about the difference between conjugacy and semi-conjugacy). Here are some first questions:

1. Is the space nontrivial?
   (i) Does Γ act nontrivially (or faithfully) on S^1?
   (ii) as a more refined question, we can ask: Does a given S^1 bundle admit a flat connection?
2. Can we describe:
   (i) the connected components of this space? Connected components correspond to the *deformation classes* of flat bundles (or actions, or representations) (ii) isolated points (these correspond to rigid representations).
3. Can we parameterize $\mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1))/\sim$:
   (i) Are there natural or reasonable coordinates (or even local coordinates) on this space?

A FEW EXAMPLES OF REPRESENTATIONS

- Take a surface group $\Gamma = \pi_1(\Sigma)$. Embed Γ in $\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{Homeo}_+(S^1)$

- Take a group, abelianize it, and map the abelianization to $S^1 \subset \text{Homeo}_+(S^1)$ (as rotations)
- If Γ is the free group, one can specify arbitrary homeomorphisms as the images of the generators; this defines a representation.
- A more sophisticated example: if M^3 has a pseudo-Anosov flow, then there is a faithful "universal circle" representation $\pi_1(M^3) \to \text{Homeo}_+(S^1)$

3. Coordinates on Hom$(\Gamma, SL(2,\mathbb{R}))/\sim$

Since $\text{Homeo}(S^1)$ is complicated, let's look at an easier space – $\text{Hom}(\Gamma, SL(2,\mathbb{R}))/\sim$ (now $\sim$ really means conjugacy).

$\text{Hom}(\Gamma, SL(2,\mathbb{R}))/\sim$ has (well-known) *trace coordinates*. Basic facts about trace are:

- A function to $\mathbb{R}$
- It is a conjugation invariant, $\text{tr}(ghg^{-1}) = \text{tr}(h)$
- It is not a homomorphism

The fact that trace gives coordinates is the following theorem:

THEOREM 3.1.

Given two nondegenerate representations, $\rho_1, \rho_2 \in \text{Hom}(\Gamma, SL(2,\mathbb{R}))$, if $\text{tr}(\rho_1(\gamma)) = \text{tr}(\rho_2(\gamma)) \forall \gamma \in S$ where S is a certain finite subset of Γ, then $\rho_1 \sim \rho_2$.

Now we want to imitate this process for $\text{Hom}(\Gamma, \text{Homeo}_+(S^1))/\sim$. We'll look for a conjugation-invariant function $\text{Homeo}_+(S^1) \to \mathbb{R}$.

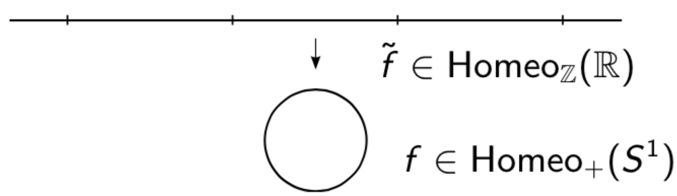

Figure 3: Poincaré, universal cover of circle

DEFINITION 3.2. (the Poincaré *translation number*) Given f in $\text{Homeo}(S^1)$, pick a lift $\tilde{f}$ of f to a homeomorphism of the line, and define

$$\tau(\tilde{f}) = \lim_{n\to\infty} \frac{\tilde{f}^n(p_0)}{n}$$

Intuitively, the translation number captures the average distance $\tilde{f}$ translates points along the line.

Much like trace, this is conjugation-invariant (in fact it is invariant under semi-conjugacy), and not a homomorphism.

Exercise.

Find $\tilde{f}, \tilde{g}$ with $\tau(\tilde{f}) = \tau(\tilde{g}) = 0$ but $\tau(\tilde{f}\tilde{g}) = 1$.

Problem.

This does not define a function on $\mathrm{Homeo}(S^1)$ since τ depended on our choice of the lift.

There are two solutions to this problem

(i) Look at τ mod $\mathbb{Z}$, i.e. $\lim_{n\to\infty} \frac{\tilde{f}^n(0)}{n}$ mod $\mathbb{Z}$. This does not depend on the lift, so τ mod $\mathbb{Z}$: $\mathrm{Homeo}_+(S^1) \to \mathbb{R}/\mathbb{Z}$.

(ii) Alternatively, we can define a "cocycle" $c(f,g) = \tau(\tilde{f}\tilde{g}) - \tau(\tilde{f}) - \tau(\tilde{g})$, it is easy to check that $c(f,g)$ does not depend on the choice of lifts $\tilde{f}$ and $\tilde{g}$.

Remark.

τ mod $\mathbb{Z}$ does not give coordinates. For example, take $\Gamma = \pi_1(\Sigma_g)$ and the Fuchsian representation $(PSL(2,\mathbb{R}))$. Then τ mod $\mathbb{Z}(\rho(\gamma)) = 0$ for all γ, which is the same as for the trivial representation!

So if we want to define coordinates, we should look at c instead.

4. A cocycle

$c(f,g)$ satisfies the cocycle condition (exercise!), in other words $[c] \in H_b^2(\mathrm{Homeo}_+(S^1);\mathbb{R})$. (for the experts: the translation number is a quasimorphism... this is the standard way to turn a quasimorphism into an element of second bounded cohomology)

Given $\rho : \Gamma \to \mathrm{Homeo}_+(S^1)$, use representation ρ to pullback $\rho^*[c] \in H_b^2(\Gamma;\mathbb{R})$.

Theorem 4.1 (Ghys, Matsumoto).

$\rho \in \mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1))/\sim$ is determined by $\rho^*[c] \in H_b^2(\Gamma,\mathbb{R})$, and the value of $\tau_{\mathrm{mod}\mathbb{Z}}(\rho(\gamma))$ on any set of generators for Γ.

Hence, the translation numbers captures the space completely, and can be thought of as giving "coordinates". This perspective I learned from Danny Calegari.

5. Applications of "rotation number coordinates"

The study of translation numbers plays an important role in the proofs of the following theorems:

1. Milnor-Wood [Wo]: (Existence of flat connection)
   $S^1 \to E$ admits a flat connection $\iff$ |Euler number| $\leq |\chi(\Sigma)|$
   $\downarrow$
   Σ

the Euler number comes from a the characteristic class of bundle. Wood's proof makes essential use of understanding the translations of lifts of homeomorphisms.

2. Matsumoto [Mat87]: (Rigidity) $\rho : \pi_1(\Sigma) \to \text{Homeo}_+(S^1)$ has maximal Euler number $\iff$ semi-conjugate to Fuchsian representation.

3. Calegari [Ca]: (Rigidity) Calegari uses Matsumoto's ideas and a study of rotation numbers to give examples of groups Γ with few or only rigid actions on S^1.

4. Calegar-Walker [CW]: Given free group F, describe "slices" of $\text{Hom}(F, \text{Homeo}_+(S^1))$ in *translation number coordinates.* The figures below (drawn by Calegari and Walker) illustrate the possible translation numbers of a word in the generators of F as compared to the translation numbers of the generators.

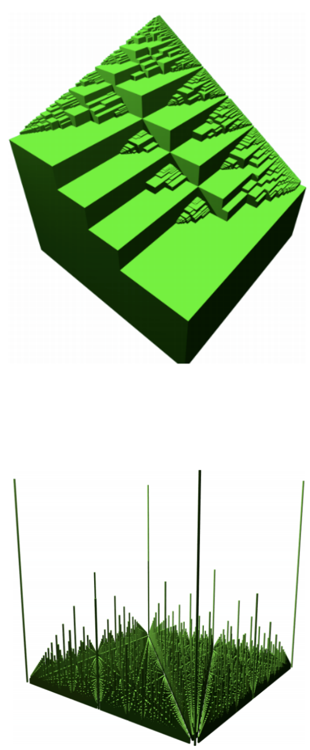

Figure 4: From Calegari, Walker *Ziggurats and rotation numbers* [CW]

6. Applications, Cont.

1. Mann [Man14]: (Connected components and rigidity)
Using techniques from Calegari– Walker, I found

(i) New examples of rigid representations $\pi_1(\Sigma_g) \to \mathrm{Homeo}_+(S^1)$ with non-maximal Euler number.

(ii) Identification/classification of more connected components of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{Homeo}_+(S^1))/\sim$.

The new rigid representations are constructed as follows: take a Fuchsian representation. If the Euler number of this is diviisble by k, the representation lifts to the k-fold cover of the circle. Any such lift (there are k^{2g} of them) will give a rigid representation.

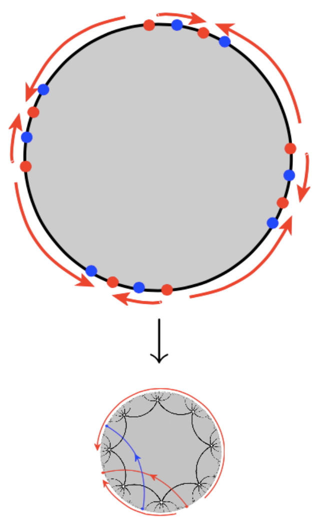

Figure 5: lift to k-fold cover of S^1

7. Open Questions

1. Does $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(S^1))/\sim$ have infinitely many connected components?

(i) We have that $\mathrm{Hom}(\pi_1(\Sigma), PSL(2,\mathbb{R}))/\sim$ has finitely many, classified by Goldman [Go].

2. Are there more examples of rigid representations in

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(S^1))/\sim\ ?$$

3. Is the space of foliated $S^1 \times \Sigma$ products connected? (flat bundles with Euler number 0)
4. Is $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(S^1))$ locally connected?
5. How about groups other than $\pi_1(\Sigma)$?

8. Another Perspective

We examine the question of whether Γ acts nontrivially/faithfully on S^1.

Theorem 8.1.

Γ acts faithfully on $\mathbb{R} \iff \Gamma$ is *left-orderable*. *Left-orderable* means that there exists a total order on Γ satisfying $a < b \iff ga < gb$.

For example, $\mathbb{R}$ (as an additive group, with the usual order) is left-orderable.

Application (Witte Morris [Mo]).

If $\Gamma < SL(n, \mathbb{Z})$ is finite index (and $n \geq 3$), then Γ has no faithful action on S^1.

Here is the idea of the proof: Using a theorem of Ghys, show that (a finite index subgroup of) Γ fixed a point in the circle. This gives an action on the line $\mathbb{R}$. Now show that the (finite index subgroup of) Γ is not left-orderable – contradiction!

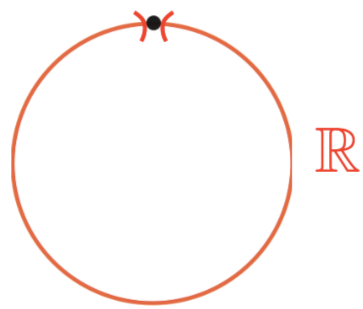

Figure 6: A fixed point on the circle gives an action on the line

Remark.

It is an open question if $\Gamma < SL(n, \mathbb{R})$ lattice for $n \geq 3$ has a faithful action on S^1, or if all actions factor through a finite quotient.

Many partial/related results are known (see references in [Mo]).

Let's return to our original question – trying to find an algebraic characterization of groups that act faithfully on the circle, similar to left orderable groups acting on the line. We want to define "circular orderability" so that the following theorem is true:

Theorem 8.2.

Γ acts faithfully on $S^1 \iff \Gamma$ is circularly-orderable.

Remark: whatever "circluarly orderable" means, S^1 should be an example.

How to capture the "order" of points on S^1? Comparing two points doesn't make sense, but for 3 points, we can say $x < y < z$ in terms of their counterclockwise position on the circle. In other words, it makes sense to say that the ordered triple (x, y, z) is positively or negatively oriented. Also, this orientation of triples is left-multiplication invariant.

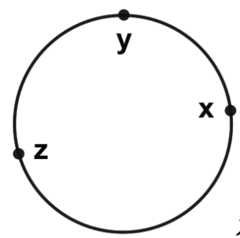

Figure 7: Circularly-orderable

9. Circular Orders

Definition 9.1. We define a *circular order* on Γ to be a function $ord : \Gamma \times \Gamma \times \Gamma \to \{\pm 1, 0\}$ that satisfies a compatibility condition on 4-tuples. It maps

$$(x, x, y) \mapsto 0,$$
$$(x, y, z) \mapsto \pm 1 \text{ according to the orientation of the triple.}$$

Exercise.

What is the compatibility condition on 4-tuples?

Hint: The "compatibility condition" on 4-tuples is the *cocycle condition!* (for inhomogeneous cocyles). In other words,

$$[ord] \in H_b^2(\Gamma; \mathbb{Z})$$

The following theorem turns out to be true

Theorem 10.1.

(Recall) If Γ has circular-order,

$$\exists \text{ a faithful } \rho : \Gamma \to \mathrm{Homeo}_+(S^1).$$

Theorem 10.2 (Thurston, Ghys, ...).

$$[ord] = 2\rho^*[c] \text{ in } H_b^2(\Gamma;\mathbb{R}).$$

10. Homework Exercise

- Describe the actions of your favorite group Γ on S^1.
- There are many interesting "geometric" examples to consider: Γ = lattice in semi-simple Lie group, $\Gamma = \pi_1(M^3), \Gamma = MCG(\Sigma_{g,*}), \Gamma = MCG(\Sigma_{g,b}), \pi_1(\Sigma_g)$,etc.

11. Epilogue (not covered in talk)

Other perspectives on group actions on the circle not mentioned yet:

- Semi-conjugacy versus conjugacy. (nice intro in [BFH]) (also relates to regularity issues, see below)
- Regularity: Compare Hom(Γ,G) where $G = \mathrm{Diff}^r(S^1)$ or G = Homeo(S^1) or $G = PSL(2,\mathbb{R})$. What about $G = QS(S^1)$?

  (Goldman [Go] for $PSL(2,\mathbb{R})$, Bowden [Bo] and Navas [Na] for Diff^r, Ghys)
- Many other perspectives on bounded cohomology, e.g. continuous bounded cohomology, and applications to actions on S^1 ([Bu] and references there)
- Tools from low dimensional dynamics, often applicable in higher regularity case. In Homeo case, new ideas in [Mat14] may be promising.
- This talk focused on Γ... but can we understand $\mathrm{Homeo}_+(S^1)$ better as a group? How to think of it as an *infinite dimensional Lie group*? What about $\mathrm{Diff}_+(S^1)$ (truly a ∞-dimensional Lie group)? What is the algebraic structure of these groups, and how does it relate to their topological structure? (see e.g. [Man15])

12. References and Recommended Reading

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