# An Introduction to Lorentzian 3-manifold

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## 1. Lorentzian Geometry

DEFINITION 1.1. We define a *Lorentzian vector space* $V \in \mathbb{R}^{2,1}$ as a 3-dimensional real vector space with *Lorentzian inner product*, a symmetric, non-degenerate bilinear form of signature $(2,1)$.

Consider the system given by $u \cdot v = u_1v_1 + u_2v_2 - u_3v_3$, that with the standard basis $\{e_1, e_2, e_3\}$ may be represented by $\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$.

### 1.1 Vectors in Lorentzian Vector Spaces

A vector $v \neq 0 \in V$ is called

- *timelike* if $v \cdot v < 0$
- *lightlike* (null) if $v \cdot v = 0$
- *spacelike* if $v \cdot v > 0$.

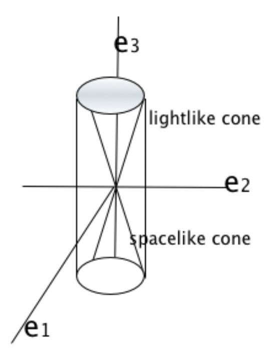


Figure 1: Diagram of spacelike and timelike cones

We define the *Lorentz-orthogonal space* $v^{\perp}$ as the 2-dimensional linear subspace given by $\{u \in \mathbb{R}^{2,1} | u \cdot v = 0\}$.

- For *timelike* $v$, we have that $v^{\perp}$ is positive definite.
- For *lightlike* $v$, we have that $v^{\perp}$ is tangent to the null cone.
- For *spacelike* $v$, we have that $\{s, s^{-}, s^{+}\}$ forms the null-basis and we see that $v$ is a spine as shown in Fig. 2.

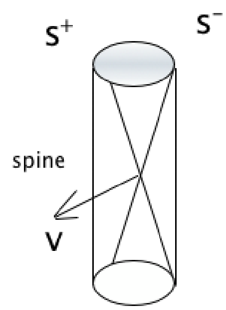


Figure 2: spacelike v

### 1.2 Properties

Definition 1.2 (Time Orientation). We define a non-spacelike vector v to be *future-pointing* if its third component $v_3 > 0$, and it v is *past-pointing* otherwise ($v_3 \leq 0$).

Definition 1.3 (Cross Product). We relate the determinant of a matrix formed by column vectors $u, v, w$ with the cross product,

$$\text{Det}(u, v, w) = u \times v \cdot w.$$

Example.

For null basis $\{s, s^{-}, s^{+}\}$,

$$s \times s^{-} = s^{-},$$
$$s \times s^{+} = -s^{+}.$$

### 1.3 Minkowski Space

We let E be the affine space modeled on $\mathbb{R}^{2,1}$. E is also a smooth, oriented manifold with a Lorentzian (semi-Riemannian) metric, and it is oriented since $\mathbb{R}^{2,1}$ is oriented.

Considering its time orientation, it is additionally defined as a flat *Lorentzian manifold*.

EXAMPLE.

$\mathbb{H}^2_{\mathbb{R}} = \{$v: future-pointing$|v{\cdot}v = -1\}$ (or $\{$timelike lines$\}/\sim$), where $\cosh(\rho(u,v)) = u \cdot v$. We can see that this hyperbolic plane sits in Minkowski space.

### 1.4 Linear Lorentzian Isometries

We give a few examples of Linear Lorentzian Isometries,

- $O(2,1)$: Group of linear transformations that preserves $B = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$

$$= \{A \in GL(3,\mathbb{R}) | A^T BA = B\}.$$

- $SO(2,1) : O(2,1) \cap GL^+(3,\mathbb{R})$, which has positive determinant if and only if it is orientation-preserving.
- $\text{Isom}^+(E) =$ orientation-preserving isometries of E.

REMARKS.

(i) $O(2,1)$ has four components. $SO^0(2,1)$ is the identity component; it is orientation and time-orientation preserving.

For example, for a spacelike vector s, $\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$ is in the null basis $\{s, s^-, s^+\}$ of s. We may see this in Fig. 3.

(ii) *Spine reflection*: orientation-preserving but with reverse time-orientation in $\mathbb{R}$.

(iii) We explicitly choose a point $o \in E$. Any *affine transformation* can be written as

$$\gamma(p) = o + g(p - o) + \vec{v}, \vec{v} \in \mathbb{R}^{2,1}, g \in GL(3,\mathbb{R}).$$

(iv) $\gamma \in \text{Isom}^+(E) \iff g \in SO(2,1)$, where $\gamma$ is the "affine deformation of g" and $L(\gamma)$ is the "linear part of $\gamma$".

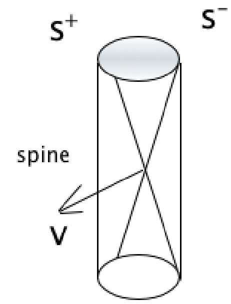


Figure 3: spacelike v

(v) $\gamma \in \mathrm{Isom}^+(E)$ is called hyperbolic, parabolic, or elliptic if $g \in SO(2,1)$ is hyperbolic, parabolic or elliptic. (g is the linear part of $\gamma$).

Example ("Hyperbolic boost").

We have $\gamma$ such that $\gamma$ is conjugate to g, where g has the representation (where $ch$ = hyperbolic cosine, $sh$ = hyperbolic sine)

g $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & cht & sht \\ 0 & sht & cht \end{pmatrix}$, for $t \in \mathbb{R}^+ \iff cht \neq 1$. Then we can write $\gamma(p) = 0 + g(p-0) + \vec{v}, \vec{v} \in \mathbb{R}^{2,1}$ where g keeps $< e_2, e_3 >$ invariant.

Case 1: g keeps $< e_2, e_3 >$ invariant. If $v \in < e_2, e_3 >$, then $\gamma|_{<e_2,e_3>}$ has a fixed point $p \in < e_2, e_3 >$.

Proof of Case 1.

$\gamma|_{<e_2,e_3>} : \begin{pmatrix} cht & sht \\ sht & cht \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

With $ch^2 t - sh^2 t = 1$ we have

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cht - 1 & sht \\ sht & cht - 1 \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ for $cht \neq 1$.

□

So we have that $\gamma(p) = p \in 0 + < e_2, e_3 >$ and therefore $\gamma$ fixes a line L passing through p and parallel to $< e_1 >$ pointwisely.

Case 2: g does not keep $< e_2, e_3 >$ invariant.

In this case, $\gamma$ has no fixed points and L is kept invariant with translations denoted by $\gamma|_L$.

This implies that $< \gamma >$ acts freely and properly discontinuously on E and $E / < \gamma > = \mathbb{R}^2 \times S^1$ is nonimpact.

## 2. Group Actions by Isometries

DEFINITION 2.1. For X a locally compact space and G a group that acts on X, G acts *properly discontinuously* on X if for every compact set $\kappa \subset X$, the set $\{\gamma \in \kappa | \gamma\kappa \cap \kappa \neq \emptyset\}$ is finite.

We say that G acts *freely* if it does not fix any points.

REMARK.

Milnor (77) asked whether a non-amenable group could act properly by affine transformations. In other words, we have the question: can a discrete free group act properly discontinuously on $\mathbb{R}^n$? By the Tit's alternative, every finitely generated linear group is either virtually solvable or contains a subgroup isomorphic to the free group of rank 2. Milnor proposed an affine deformation of a free discrete subgroup of SO(2,1) to answer this question (for example, a Schottky group).

EXAMPLE 2.2

An example of a free and properly discontinuous action is given by $E/G$, where E be the affine space modeled on $\mathbb{R}^{2,1}$ and a group G. This is a covering space, Hausdorff manifold, and a flat Lorentzian manifold.

EXAMPLE 2.3

We have linearly independent vectors $t_1.t_2, t_3 \in \mathbb{R}^{2,1}$.

$$G =< Z_1, Z_2, Z_3 >\cong \mathbb{Z}^3.$$

$Z_i : p \mapsto p + t_i$ where $Z_i$ acts freely and properly discontinuously on E.

$\Rightarrow$ $E/G$ is a compact Lorentzian manifold for E and G defined in the previous example.

Next, we look at developments in the efforts to understand "affine deformations" of a discrete group $\Gamma < O(2, 1)$; i.e., $G <$ Isom$(E)$ with $L(G) = \Gamma$.

[Fried-Goldman, 1983]: If a group G of affine transformations acts properly discontinuously on $\mathbb{R}^3$ then it is either virtually solvable or it does not act cocompactly in which case L(G) is conjugate to a subgroup of $O(2, 1)$.

[ Mess 2009]: An affine deformation of a closed surface group *cannot* act properly discontinuously on E.

## 3. Margulis Invariants

We consider an element $g \in O^0(2, 1)$, a hyperbolic group. There exists three distinct eigenvalues $(1,\ \lambda, \frac{1}{\lambda})$ such that $0 < \lambda < 1 < \frac{1}{\lambda}$. We may note that the eigenspaces of $\lambda, \frac{1}{\lambda}$ are null.

Then we have that

$$g^0 = \text{unit spacelike 1-eigenvector}$$
$$g^+ = \text{future-pointing } \tfrac{1}{\lambda}\text{-eigenvector}$$
$$g^- = \text{future-pointing } \lambda\text{-eigenvector,}$$
$$\text{and } [g^0, g^-, g^+] > 0, \text{ giving that g is positively oriented.}$$

Definition 3.1. Let $\gamma \in \mathrm{Isom}(E), L(\gamma) = g$. Then $< \gamma >$ acts freely if and only if $\forall p \in E, \gamma(p) - p \notin < g^-, g^+ >$ so there exists a unique $l_\gamma$ invariant line.

So we have that $\gamma(p) = p + \alpha g^0, \alpha \neq 0$ on $l_\gamma$. Then $\alpha = \alpha(\gamma)$ is called the *Margulis invariant* of $\gamma$.

Proposition 3.2.

$$\alpha(\gamma) = (\gamma(p) - p)g^0 \ \ \forall\ p \in E.$$

Proposition 3.3.

$$\alpha(\gamma) = 0 \text{ if and only if } \gamma \text{ fixes a point.}$$

Proposition 3.4.

$$\alpha(\gamma^{-1}) = \alpha(\gamma) \text{ (from } (g^{-1})^0 = -g^0).$$

Proposition 3.5.

$$\alpha(\eta\gamma\eta^{-1}) = \alpha(\gamma) \ \ \forall y \in \mathrm{Isom}(E).$$

Opposite Sign Lemma [Margulis]:

Let $\gamma, \eta \in \mathrm{Isom}(E)$ be hyperbolic or parabolic. If $\alpha(\gamma)\alpha(\eta) < 0$, then $< \gamma, \eta >$ *cannot* act properly on E.

Remark.

[Margulis 83] has shown there exist affine deformations of Schottky groups which act properly discontinuously on $\mathbb{R}^3$.

[Charette-Drumm] has generalized the results about Margulis invariants in the Opposite Sign Lemma for parabolic surfaces.

[Drumm 90]: Every generalized Schottky group (including parabolic groups) admits a lot of affine deformations.

Theorem 3.6.

Take group $\Gamma$ generated by $\gamma_1, ..., \gamma_n$,

$\Gamma =< \gamma_1, ..., \gamma_n >$

Then, $L(\Gamma) =< g_1, ..., g_n >$ is a Schottky group.

Suppose there exists a simply connected region $\triangle$ bounded by 2n pairwise disjoint *crooked planes* $c_1^-, c_1^+, ..., c_n^-, c_n^+$ such that $\gamma_i c_i^- = c_i^+$ for $i = 1, ..., n$. Then, $\triangle$ is a fundamental domain for $\Gamma$ and $\Gamma$ acts freely and properly discontinuously on E.