Hermitian Symmetric Spaces

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1. Symmetric Spaces

Definition 1.1. A Riemannian manifold X is a *symmetric space* if each point $p \in X$ is the fixed point set of an involutive isometry G_p of X.

Examples.

1. $\mathbb{H}^n_{\mathbb{R}}$, real hyperbolic space, is a rank 1 symmetric space.
2. $P_n = SL(n, \mathbb{R})/SO(n)$, the set of symmetric positive-definite matrices of determinant 1 with Riemannian metric $h_{\mathrm{Id}}(X, Y) = \mathrm{tr}(XY)$. We also have $\mathrm{rk}(P_n) = n-1$.

Remark.

If X is a symmetric space, then $G = \mathrm{Isom}(X)$ is transitive.

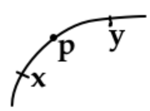

Figure 1: Inverting x and y over the midpoint

In particular, $X = G/K$. We can denote by G is isometry group of the symmetric space $X, G = \mathrm{Isom}(X)$. In addition, we denote $K = \mathrm{Stab}\ G(p)$, a compact subgroup.

Definition 1.2. The $\mathbb{R}$-*rank* of a symmetric space is the maximal n such that $\mathbb{R}^n \hookrightarrow X$ is an isometric embedding.

2. Hermitian Symmetric Spaces

Definition 2.1. A symmetric space X is *Hermitian* if it admits a G-invariant complex structure $J : TX \to TX$ with $J^2 = -\mathrm{Id}$.

Remark.

A Hermitian symmetric space X is a complex manifold. Moreover it is Kähler. A Kähler form is a closed, differentiable 2-form, $\omega(X, Y) = g(JX, Y)$.

Exercise.

Check that since X is symmetric, ω is closed.

This is an important starting point for the study of maximal representations. If G is a semisimple Lie group without compact factors,

$$H^2_{cb}(G,\mathbb{R}) = H^2_c(G,\mathbb{R}) = \Omega^2(X,\mathbb{R})^G$$
continuous bounded cohomology = continuous homology = G-invariant differential two forms on the symmetric space
$$\kappa^b_G \leftrightarrow \kappa_G \leftrightarrow \omega \text{ Kähler class}$$

EXAMPLES.

1. $\mathbb{D} = \mathbb{H}^2_{\mathbb{R}} = \mathbb{H}^1_{\mathbb{C}}$ the Poincaré disk $\{z|\bar{z}z < 1\}$

2. For $\mathbb{H}^n_{\mathbb{C}} = \{z \in \mathbb{C}^n||z| < 1\} \subset \mathbb{C}^n \subset \mathbb{CP}^n$ the complex hyperbolic spaces (family of rank 1 symmetric spaces), we consider $\mathbb{C}^{n+1}$ with the Hermitian form $h_{1,m}$ of signature $(1,n)$. Then,

$$\mathbb{H}^n_{\mathbb{C}} = \{[v] \in \mathbb{CP}^n|h|_v > 0\} = SU(1,n)/U(n) = G/K$$

where G is the connected component of the identity in isometries and K is the maximal compact subgroup of G.

3. Consider an Hermitian form $h_{m,n}$ (of signature $(m,n), m < n$) on $\mathbb{C}^{m+n}$

$SU(m,n)$ acts on $Gr_m(\mathbb{C}^{m+n})$ and we have the family of Hermitian symmetric spaces

$$\begin{aligned} X_{\mathrm{m,n}} &= \{[v] \in Gr(m,\mathbb{C}^{m+n})|h|_v \text{ is positive definite}\} \\ &= \{X \in M(n\times m)|X^*X - \mathrm{Id} < 0\} \\ &= SU(m,n)/S(U(m)\times U(n)) \end{aligned}$$

which is a rank m symmetric space, generalizing $\mathbb{H}^n_{\mathbb{C}}$.

4. The Siegel manifold, $Sp(2n,\mathbb{R})/U(n)$.

3. Geometric Features

We let X be a Hermitian symmetric space of rank r. Then, there exist totally geodesic isometric holomorphic embeddings,

$$\mathbb{D}^r \hookrightarrow X.$$

THEOREM 3.1 (HARISH CHANDRA).

Any Hermitian symmetric space can be realized as a bounded domain in a complex vector space.

EXAMPLE.

$X_{m,n}$ is biholomorphic to a bounded subset of $\mathbb{C}^{m\times n}$,

$$\{X \in M(n\times m,\mathbb{C})|X^*X - \mathrm{Id} < 0\}$$
(Hermitian form is negative definite).

Definition 3.2 (Tube type). A Hermitian symmetric space is of *tube type* if it is biholomorphic to a domain of the form $V + i\Omega$ where V is a real vector space and $\Omega \subset V$ is a properly convex open cone. We say the space is properly convex if it contains no lines, and we say it is a cone if for $v \in \Omega \to \lambda v \in \Omega \ \ \forall \lambda \in \mathbb{R}^+$ (closed under multiplication by positive reals).

Example.

We can see that the Poincaré disk, a bounded subset of the plane, has a model that is an unbounded subset, the upper half plane. The unbounded model is of the form $\mathbb{R} \times i\mathbb{R}^+$.

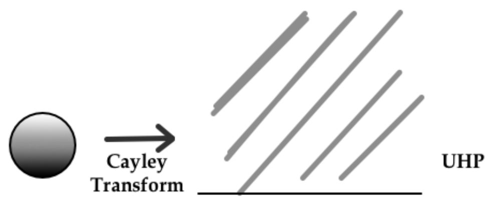

Figure 2: Tube type domain Poincaré disk to the UHP

Classifications.

Tube type	Not of tube type
$X_{m,m} = SU(m,m)/SO(m) \times U(m)$	$X_{m,n}, n > m$
$Sp(2n,\mathbb{R})/U(n)$	
$SO^*(4p)/U(2p)$	$SO^*(4p+2)/U(2p+1)$
$SO(2,n)/SO(2) \times O(n)$	
$SO(2,10)$	$E_G(-14)/SO(10) \times SO(2)$
$E_7(-25)/E_G \times SO(2)$	

4. Boundary

We consider the Hermitian symmetric space

$$X_{\text{m,n}} = \{[v] \in Gr_m(\mathbb{C}^{m+n}, h_{m,n}) | h|_v \text{ is positive definite}\}.$$

We also set the topological boundary

$$\delta X_{\text{m,n}} := \{[v] \in Gr_m(\mathbb{C}^{m+n}, h_{\text{m,n}}) | \ h_{\text{m,n}}|_v \geq 0\}.$$

The action on $X_{m,n}$ extends to a continuous action on $\delta X_{m,n}$ with a unique closed orbit that consists of maximal isotropic subspaces

$$S_{m,n} = \{[v] \in Gr_m(\mathbb{C}^{m,n}) | h|_v \equiv 0\}.$$

DEFINITION 4.1 The *Shilov boundary* is the unique closed G-orbit in the topological boundary of the bounded domain realization of X. It can be expressed as $S = G/Q$ for some parabolic subgroup Q of S.

REMARK.

If X is not of tube type and $Y \subset X$ is a maximal tube type subdomain, then $S_Y \subset S_X$. This gives a nice incidence structure on S_X when X is not of tube type.

THEOREM 4.2 (CLERC ØRSTED).

There exists a cocycle $\beta : S^3 \to \mathbb{R}$ that is the limit of area cocycles, the Bergmann cocycle.

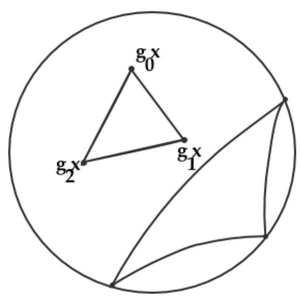

Figure 3: Bergmann cocycle

DEFINITION 4.3. The *Bergmann cocycle*

$$\beta : S^3 \to \mathbb{R}$$

is a G-invariant cocycle obtained by

$$\beta(\xi_0, \xi_1, \xi_2) = \lim_{\mathrm{x_n \to x, y_n \to y, z_n \to z}} \frac{1}{\pi} \int_{\triangle(x_n, y_n, z_n)} \omega$$

This normalizes the cocycles where ω is a Kähler form.

In Figure 3, the area in the triangle with vertices g_0x, g_1x, g_2x gives the group cocycle. We may instead give the area based on the triangle formed by taking points on the boundary of S^1.

Theorem 4.4.
The cocycle β has values in $[-\mathrm{rk}(X), \mathrm{rk}(X)]$, and:
1. If X is of tube type, then β has values in $[-\mathrm{rk}(X), \mathrm{rk}(X)] \cap \mathbb{Z}$.
2. If X is not of tube type, then it achieves all the possible values.