Introduction to Quasi-Fuchsian Groups

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(based on a lecture by Sara Maloni)

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1 Definitions and examples

DEFINITION 1.1 (FUCHS 1880, POINCARÉ 1882).

1. A discrete subgroup $\Gamma \subset \mathrm{PSL}_2\mathbb{R}$ is called *Fuchsian*.
2. When $\Gamma \cong \pi_1(\Sigma)$, we define the *Fuchsian space*

$$F(\Sigma) = \{\rho : \pi_1(\Sigma) \to \mathrm{PSL}_2\mathbb{R} | \text{discrete and faithful}\}/\mathrm{PSL}_2\mathbb{R}.$$

REMARK.

1. Γ Fuchsian $\iff$ $\Gamma \curvearrowright \mathbb{H}^2$ properly discontinuously.
2. $F(\Sigma)$ coincides with Teichmüller space $\tau(\Sigma)$ of the surface, which is the space of marked complex structures on the surface Σ, or, equivalently, the space of marked hyperbolic structures on the surface Σ.

CLASSIFICATION OF ELEMENTS OF $\mathrm{PSL}_2\mathbb{R}$.
We may classify the elements $A \in \mathrm{PSL}_2\mathbb{R}$ into 3 types:

- elliptic $\iff$ there is one fixed point in $\mathbb{H}^2$ $\iff$ $0 \le tr^2(A) < 4$.
- parabolic $\iff$ there is one fixed point in $\partial\mathbb{H}^2 = \hat{\mathbb{R}}$ $\iff$ $tr^2(A) = 4$.
- hyperbolic $\iff$ there are 2 fixed points in $\partial\mathbb{H}^2 = \hat{\mathbb{R}}$ $\iff$ $tr^2(A) > 4$.

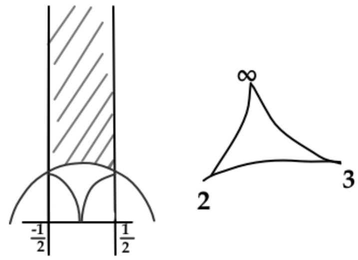

Figure 1: On the left, 3 copies of the fundamental domain for $\Gamma = \mathrm{PSL}_2\mathbb{Z}$. On the right, $\mathbb{H}^2/\Gamma$.

EXAMPLES.

1. $\Gamma = \mathrm{PSL}_2\mathbb{Z} =< \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} >$. Then $\mathbb{H}^2/\Gamma$ is an orbifold homeomorphic to $\Sigma_{0,3}$, where we have a puncture (or cusp of order ∞) and two cusp points of order 2 and 3, as shown in Figure 1.

2. $\Gamma =< \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} >\cong \mathbb{F}_2$. Then $\mathbb{H}^2/\Gamma \cong \Sigma_{0,3}$. One generator acts as shown by the red arrow in Figure 2, while the second generator acts as shown by the blue arrow in Figure 2.

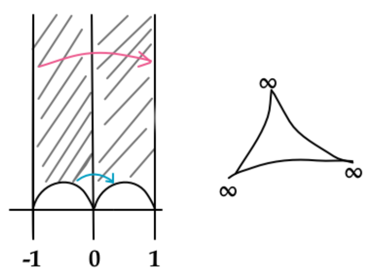

Figure 2: On the left, fundamental domain for $\Gamma \cong \mathbb{F}^2$, as described in the Example 2 above. On the right, $\mathbb{H}^2/\Gamma$.

DEFINITION 1.2 (QUASICONFORMAL).
A maps $f : D \to D'$ between domains in $\mathbb{C}$ is *quasiconformal* if it sends small circles

into small ellipses with bounded ratio of axes. More precisely, it is a diffeomorphism $f : D \to D'$ between domains in $\mathbb{C}$ with bounded dilatation K_f, where

- $f_z = \frac{1}{2}(f_x - if_y)$;
- $f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$;
- $D_f := \frac{|f_z|+|f_{\bar{z}}|}{|f_z|-|f_{\bar{z}}|}$;
- $K_f = \min_{z\in D} D_f(z)$.

So f is quasiconformal if $|K_f| < \infty$ (and f conformal if $K_f = 1$).

Definition 1.2 (Klein 1863, Poincaré 1883)
A discrete subgroup $\Gamma \subset \mathrm{PSL}_2\mathbb{C}$ is called *Kleinian*.

Remark.
Kleinian group Γ will act properly discontinuously on $\mathbb{H}^3$. On the other hand, $\Gamma \curvearrowright \hat{\mathbb{C}} = \partial\mathbb{H}^3$, but it will not act properly discontinuously on the whole $\hat{\mathbb{C}}$. So we have the following decomposition of $\hat{\mathbb{C}}$.

Definition 1.3 (Domain of discontinuity, limit set).

- The *limit set* is $\Lambda(\Gamma) = \{\text{accumulation points of } \Gamma\text{-action}\}$.
- The *domain of discontinuity* is $\Omega(\Gamma) = \{x \in \hat{\mathbb{C}} | \Gamma\text{-action is properly discontinuous}\}$

Classification of elements of $\mathrm{PSL}_2\mathbb{C}$.
We may classify the elements $A \in \mathrm{PSL}_2\mathbb{C}$ into 3 types:

- elliptic $\iff$ there is one geodesic γ in $\mathbb{H}^3$ fixed pointwise and A rotates other points around $\gamma \iff 0 \leq tr^2(A) \in [0, 4)$.
- parabolic $\iff$ there is one fixed point in $\partial\mathbb{H}^3 = \hat{\mathbb{C}} \iff tr^2(A) = 4$.
- hyperbolic $\iff$ there is one fixed geodesic γ in $\mathbb{H}^3$ and A translates and rotates points around $\gamma \iff tr^2(A) \notin [0, 4]$.

Examples.

1. *Schottky group*: We can choose $g \geq 2$ pairs or mutually disjoint circles with mutually disjoint interiors $\{C_1, C_1', ..., C_n, C_n'\}$ in $\mathbb{C}$. Now, for each i, we consider element $A_i \in \mathrm{PSL}_2\mathbb{C}$ such that

- $A_i(C_i) = C_i'$;
- $A_i(\mathrm{Int}(C_i)) = \mathrm{Ext}(C_i')$,

that is we find elements in $\mathrm{PSL}_2\mathbb{C}$ which maps circles to circles and the exteriors of circles into the interiors of other circles. Then the free group $\Gamma =< A_1, \ldots, A_g >$ is Kleinian.

2. *Quasi-Fuchsian group*: It is a Kleinian group $\Gamma \subset \mathrm{PSL}_2\mathbb{C}$ such that limit set $\Lambda(\Gamma)$ is a quasi-circle. They are obtained as quasiconformal deformation of a Fuchsian groups. For a surface Σ, we can then define the *quasi-Fuchsian space*

$$\mathcal{QF}(\Sigma) = \{\rho : \pi_1(\Sigma) \to \mathrm{PSL}_2\mathbb{C} | \rho(\pi_1\Sigma) \text{ is quasi-Fuchsian}\}/\mathrm{PSL}_2(\mathbb{C}).$$

2 Quasi-Fuchsian Groups

Given $\rho \in \mathcal{QF}(\Sigma)$, let $\Gamma_\rho = \rho(\pi_1\Sigma)$ and

$$M_\rho = \mathbb{H}^3/\Gamma_\rho \cong \Sigma \times \mathbb{R},$$

the associate 3–manifold, and

$$\overline{M}_\rho = (\mathbb{H}^3 \sqcup \Omega(\Gamma_\rho))/\Gamma_\rho \cong \Sigma \times [0,1],$$

its compactification.

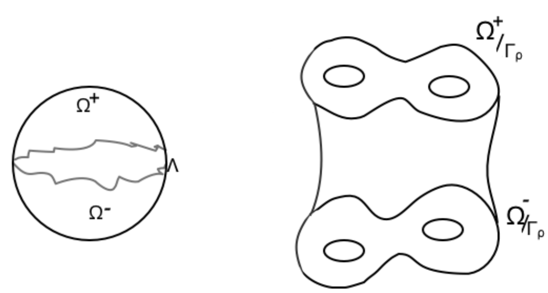

Figure 3: On the left, the limit set of a quasi-Fuchsian group. On the right the compact manifold $\overline{M}_\rho$.

Definition 2.1. (Convex core) Given $\rho \in \mathcal{QF}(\Sigma)$, the *convex core* C_ρ of the hyperbolic 3–manifold M_ρ is the smallest non-empty convex subset of the 3–manifold M_ρ such that the inclusion is a homotopy equivalence. Hence it is the smallest convex subset which carries all the fundamental groups. Another definition is $C_\rho = CH(\Lambda_\rho)/\Gamma_\rho$, where $CH(\Lambda_\rho)$ is the convex hull of Λ_ρ.

Theorem 2.2 (Thurston).
Given $\rho \in \mathcal{QF}(\Sigma)$, the boundary ∂C_ρ of the convex core of M_ρ is a pleated surface.

Definition 2.3 (Laminations)

- A *geodesic lamination* γ is a closed set of pairwise disjoint complete simple geodesics on S.
- A *transverse measure* on γ is a measure on the arcs transverse to the leaves of γ invariant under pushforward maps.
- The space of *measured laminations* $ML(\Sigma)$ on Σ is the set of pairs (λ, μ), where γ is a geodesic lamination and μ is a transverse measure on γ.

Definition 2.3

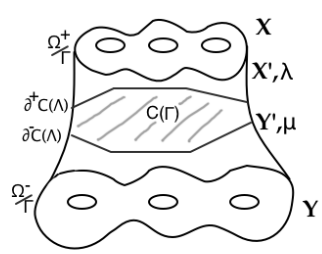

Figure 4: A quasi-Fuchsian manifold with the associated convex core.

Definition 2.4 (Pleated surface) A *pleated surface* in a hyperbolic 3-manifold is a surface which is totally geodesic almost everywhere and such that the locus of points where it fails to be totally geodesic is a geodesic lamination. They are almost polyhedral surfaces.

Remarks.

- We can define *complex Fenchel-Nielson coordinates* on quasi-Fuchsian space $\mathcal{QF}(\Sigma)$
$$FN_{\mathbb{C}} : \mathcal{QF}(\Sigma) \to (\mathbb{C}_+ \times \mathbb{C})^{3g-3}.$$
This maps is not surjective.

- We can define *Dehn-Thurston coordinates* on the measured lamination space $ML(\Sigma)$ $DT : ML(\Sigma) \to (\mathbb{R}_+ \times \mathbb{R})^{3g-3}$.

Theorem 2.6 (Bers' Simultaneous Uniformization Theorem). Any quasi-Fuchsian group $\rho \in \mathcal{QF}(\Sigma)$ is uniquely determined by the 2 Riemann surfaces $\Omega_\rho^+/\Gamma_\rho$ and $\Omega_\rho^-/\Gamma_\rho$. So quasi-Fuchsian space $\mathcal{QF}(\Sigma)$ can be parameterized by the product of 2 copies of Teichmüller space:
$$\mathcal{QF}(\Sigma) \cong \tau(\Sigma) \times \tau(\overline{\Sigma})$$
$$\rho \mapsto (\Omega_\rho^+/\Gamma_\rho, \Omega_\rho^-/\Gamma_\rho).$$
The theorem is saying that it is possible to simultaneously uniformize any 2 different Riemann surfaces of the same genus using a quasi-Fuchsian group.

Bending Conjecture.

Given a quasi-Fuchsian group $\rho \in \mathcal{QF}(\Sigma)$, let $(X, Y) \in \tau(\Sigma) \times \tau(\overline{\Sigma})$ be the conformal structures on Ω_ρ/Γ_ρ, let $(X', Y') \in \tau(\Sigma) \times \tau(\overline{\Sigma})$ be the conformal structures on ∂C_ρ, and let $(\lambda, \mu) \in ML(\Sigma) \times ML(\Sigma)$ be the pair of *admissible* measure laminations on the pleated surface C_ρ. See Figure 5. The conjecture says that $\mathcal{QF}(\Sigma)$ can be parametrized by the pairs (X', Y'), or by (λ, μ):
$$\mathcal{QF}(\Sigma) \cong ML(\Sigma) \times ML(\Sigma)$$
$$\rho \mapsto (\lambda, \mu).$$
$$\mathcal{QF}(\Sigma) \cong \tau(\Sigma) \times \tau(\overline{\Sigma})$$
$$\rho \mapsto (X', Y').$$
Bonahon showed, using the bending and shearing cocycles, that these two definitions are equivalent.

If we are bending along simple closed curves, and the laminations are rational, this conjecture is true (use Hodson and Kerckhoff's rigidity of cone manifolds). The conjecture is also know for the case of the once punctured torus $T_{1,1}$.

3 Slices of $\mathrm{AH}(\Sigma)$

Let $\mathrm{AH}(\Sigma) = \{\rho : \pi_1\Sigma \to \mathrm{PSL}_2\mathbb{C} | \text{discrete and faithful}\}/\mathrm{PSL}_2\mathbb{C}$.

We study this space since its interior is the quasi-Fuchsian space, that is

$$Int(\mathrm{AH}(\Sigma)) = \mathcal{QF}(\Sigma).$$

Bers' Double Uniformization Theorem tells us that $\mathcal{QF}(\Sigma)$ is homeomorphic to a ball.

Some examples of linear slices include:

- *Bers slice*: Using Bers' double uniformization theorem, we can consider the quasi-Fuchsian manifolds where we fix the top (or bottom) conformal structure of the boundary. These groups form a slice in quasi-Fuchsian space, homeomorphic to the Teichmüller space $\tau(\Sigma)$, which is the Bers embedding of the Teichmüller space into quasi-Fuchsian space.

- *Maskit Slice*: It is similar to a Bers slice, but, instead of fixing a point in Teichmüller space, one fixes a measured lamination. It is a slice in the boundary $\partial \mathrm{AH}(\Sigma)$, so these groups are no longer quasi-Fuchsian.

4 Topology of $\mathrm{AH}(\Sigma)$

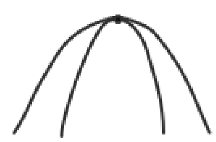

Figure 5: self-bumping

- *Self-bumping* (McMullen): A point ρ is a *self-bumping* point if there is a neighborhood N of ρ such that, for every sub-neighborhood N', the intersection $N' \cap \mathrm{AH}(\Sigma)$ is disconnected.

- *Non-local connectivity* (Bromberg, Magid): $\mathrm{AH}(\Sigma)$ is not locally connected.

- *Local connectivity at generic points* (Broch-Bromberg-Canary-Minsky): At conformally rigid points, $\mathrm{AH}(\Sigma)$ is locally connected.

5 Three Big Conjectures

5.1 Density

THEOREM 3.1.
$\mathrm{AH}(\Sigma) = \overline{\mathcal{QF}(\Sigma)}$.

This theorem tells us that every finitely generated Kleinian group is a limit of a geometrically finite Kleinian group.

Brock (2009) proved the conjecture made by Bers on simply degenerate groups in the boundary of the Bers' slice. Then, Brock and Bromberg (2004) proved the conjecture for freely indecomposable groups without parabolic elements. Finally, Namazi-Souto (2010), and Oshika (2011), independently, proved the conjecture.

5.2 Tameness

DEFINITION 3.2

- A manifold is *topologically tame* if it is homeomorphic to the interior of a compact 3–manifold.
- A manifold is *geometrically tame* if each end is either geometrically finite or simply degenerate.

THEOREM 3.2.
Every complete hyperbolic 3–manifold with finitely generated fundamental group is topologically tame.

It was conjectured by Marden (1974), and proved by Agol (2004) and Calegari–Gabai (2004), independently.

Marden proved it for geometrically finite groups.

Bonahon proved it for manifolds with freely indecomposable fundamental group, that is, manifolds with incompressible boundary. He proved this by introducing the notion of *geometrically tame*.

Canary proved that being geometrically tame is equivalent to being topologically tame.

It implies Ahlfors' measured conjecture.

5.3 Ending Lamination Theorem

THEOREM 3.4 (ELT) Every hyperbolic 3–manifold with finitely generated fundamental group is determined, up to isometry, by its end invariants.

Broch-Canary-Minsky proved this theorem for a surface group, and one can use the Tameness theorem to conclude.