Harmonic Maps and Surface Group Representations

Qiongling Li
Notes by Serena Yuan

I. Harmonic Maps to Smooth Manifolds

1. Basics

Let $u : (X, g) \to (Y, h)$ be a C^∞ map between Riemannian manifolds. We define the energy of the map u by energy function

$$E(u) = \int_X |du|^2 dvol$$

for $du \in \Gamma(T^*X \bigotimes f^*T^*Y)$. We have that the energy is conformally invariant.

Definition 1.1. A map u is said to be *harmonic* if it minimizes the energy $E(u)$ among $C'(X, Y)$.

Definition 1.2. The *Euler-Lagrange equation* is $\triangle u^k + g^{ij}\Gamma^k_{ij}u^i u^j = 0$ where $g^{ij}\Gamma^k_{ij}$ is the derivate of h.

Example.

Consider $u : \mathbb{R}^2 \to \mathbb{R}$ with the usual metric. Then the Euler-Lagrange equation reduces to $\triangle u = 0 (\triangle = \delta^2/\delta x^2 + \delta^2/\delta y^2)$.

In particular, take the example that is given by $u(x, y) = y; u : (x, y) \mapsto y$. In this example, u contracts all leaves $\{x = \text{constant}\}$.

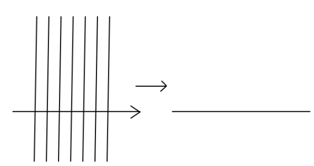

Figure 1: $u : (x, y) \mapsto y$

Theorem 1.1. (Eells-Sampson, Hartman)

Let M, N be compact, Riemannian manifolds and $K_N \leq 0$ with a given continuous map $f : M \to N$. Then there exists a harmonic map $M \to N$ that is homotopic to f. Moreover, provided f does not map onto a geodesic, then this harmonic map is unique.

In the setting, we have $id : (S, \sigma) \to (S, \rho)$ for compact surfaces of genus $g \geq 2$ with hyperbolic metrics.

Theorem 1.2.

There exists a unique harmonic map: $(S, \sigma) \to (S, \rho)$ that is homotopic to the identity.

2. Harmonic Map Description of Teichmüller Theory

We identify $(S, \sigma) = \Sigma$ as a Riemann Surface. A quadratic differential is a section of $T^*\Sigma^{1,0} \bigotimes T^*\Sigma^{0,1} \phi dz^2$. We will denote the set of all holomorphic quadratic differentials by $QD(\sigma)$.

Definition 1.3. Given $u : (S, \sigma) \to (S, \rho)$, we may associate a quadratic differential, called the *Hopf differential*,

$$\phi_u(\rho) = (u^*\rho)^{2,0} = (\rho(\tfrac{\delta u}{\delta z}, \tfrac{\delta u}{\delta z})dz^2)$$
$$\text{(the } dz \bigotimes dz \text{ part)}$$

Remark.

ϕ_u is holomorphic $\iff$ u is harmonic.

Definition 1.4.

We give the following associations,

$$M_{-1}(S) = \{\text{all hyperbolic metrics on S }\}$$
$$\text{Diffeo}(S) := \{ \text{ diffeomorphisms isotopic to the identity } \}$$
$$\text{with Diffeo}(S) \hookrightarrow M_{-1}(S) \text{ by pullbacks.}$$
$$\text{Teichmüller space } \tau(S) := M_{-1}(S)/\text{Diffeo}(S)$$

Theorem 1.5 (Teichmüller's Theorem).

$$\tau(S) \cong \mathbb{R}^{6g-6}$$

Proof. We use harmonic map theory to give a proof of the theorem.

For each $\sigma \in \tau(S)$, we have the map

$$H_\sigma : \tau(S) \to QD(\sigma)$$
$$\rho \mapsto \phi_{U(\rho)}$$

(Hopf differential $\phi_u(\rho)$ of the unique harmonic map $(S, \sigma) \to (S, \rho) \sim id$, homotopic to the identity.)

We can define $\text{Hopf}(u) = (u^*\rho)^{2,0} \in T^*X^{1,0} \bigotimes T^*X^{1,0}$ where u harmonic implies that Hopf(u) is holomorphic. So we have a map $H_\sigma : \tau(S) \to QD(\rho)$, $\rho \mapsto \text{Hopf}(U_\rho)$.

Theorem 1.6 (Wolf).

H_ρ is a diffeomorphism.

3. Relation to Representations

Considering the Riemannian universal cover of (S, ρ), we have that

$$\tau(S) \subset \chi(\pi, PSL(2,\mathbb{R})) = \mathrm{Hom}(\pi_1(S), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R}).$$

We may conclude that

$$\forall \rho \in \chi(\pi, PSL(2,\mathbb{R})) \text{ we can give } (S,\sigma) = \Sigma.$$

Then there exists a unique ρ-equivalent harmonic map, $\widetilde{\Sigma} \to \mathbb{H}^2$ where $\widetilde{\Sigma}$ is a holomorphic quadratic differential on Σ.

Remarkably, Hitchin, Simpson, Donaldson, and Corlette found that this is the base case of a profound correspondence, $\rho \in \chi(\pi_1, G)$ where G is a semi-simple Lie group and ρ has Zariski dense image.

Given Σ and G/K symmetric space for G, there exists a unique ρ-equivalent harmonic map: $\widetilde{\Sigma} \to G/K$ which give the holomorphic objects over Σ called the *Higgs bundle.*

Examples.

For $G = PSL(n,\mathbb{C}), PSL(n,\mathbb{R})$ we have $(q_2, q_3, ..., q_n)$ and for $G = Sp(2n,\mathbb{R})$ we have $(q_2, q_4, ..., q_{2n})$.

II. Harmonic Maps to Singular Spaces

1. Basics

Definition 2.1 *Singular spaces* do not refer to smooth manifolds but to metric spaces.

Examples.

- For example, a tripod.
- a tree
- $\mathbb{R}$-tree (may be locally infinite). An example is $(\mathbb{R}^2, d(p,q) = |p| + |q|)$.
- $\mathbb{R}$-buildings. They are generalizations of $\mathbb{R}$-trees where we can replace geodesics $\mathbb{R}$ by $(\mathbb{R}^n$, usual metric$)$.

We examine the question of how to give a definition of harmonic maps into singular spaces. We have $u : X \to Y = M^m \to Y$.

Recall if Y is smooth then $E(u) = \int_M |du|^2 dvol|_M$. Replace $|du|^2$ by an ϵ-approximation. $e_\epsilon(x) = \int_{\delta B_\epsilon(x)} \frac{d^2(u(x),u(y))}{\epsilon^2} \frac{d\sigma(y)}{\epsilon^{m-1}}$ where ϵ^2 is the square ratio between distances in M and Y, the stretch.

So the energy $E(u) = \text{ lim sup}_{\epsilon \to 0} \int_M e_\epsilon(x)dx$

If u is smooth, then $\lim_{\epsilon \to 0} e_\epsilon(x) = |du|^2$. In other words, harmonic maps are the ones that minimize energy.

Remark.

Harmonic maps are also locally harmonic.

2. How Harmonic Maps Relate to Representations

We now examine how harmonic maps to a singular space help us to understand representations. We are able to use that harmonic maps to singular spaces can appear as limits of harmonic maps to smooth manifolds. The answer to this question is compactification. In the $PSL(2, \mathbb{R})$ case:

$$\tau(S) \overset{\mathrm{H}_\sigma}{=} QD(\sigma).$$

Define a norm on $QD(\sigma)$ by $||\phi|| = \int_S |\phi(z)| dvol_S$

Let $BQD(\sigma) = \{\phi \in QD(\sigma) : ||\phi|| < 1\}$

$SQD(\sigma) = \{\phi \in QD(\sigma) : ||\phi|| = 1\}$

$\overline{BQD}(\sigma) = BQD(\sigma) \cup SQD(\sigma)$.

Then consider a map $\widehat{H} : \tau(S) \to BQD(\sigma)$ mapping $\rho \mapsto \frac{4H_\rho(\rho)}{1+4||H_\sigma(\rho)||}$

Since $\widehat{H}$ is a homeomorphism onto, we identify $\tau(S)$ with $BQD(\rho)$ and defnine a compactification $\overline{\tau}(S) = \tau(S) \cup SQD_\sigma = \overline{BQD_\sigma}$.

Definition 2.1. A measured foliation (F, μ) on a Riemann Surface is a singular foliation F with a transverse measure μ.

Example.

A quadratic differential ϕ defines a measured foliation. Away from zeros, $\phi = d\xi^2, \xi = X + iY$.

Then the leaves are equal to $\{x = \text{constant}\}$, with measure $|dx|$. We then "patch together" the leaves to give a vertical measured foliation, $F_u(\phi)$.

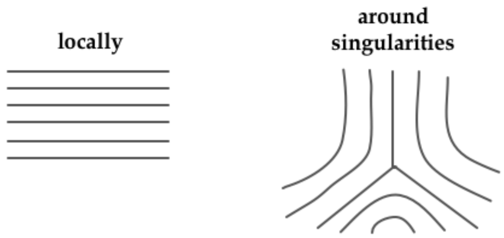

Figure 2: measured foliations

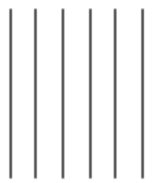

Figure 3: vertical measured foliation

Definition 2.2 (Wolf's Compactification). Wolf's compactification is given by $\overline{T(S)}^W_\rho = \tau(S) \cup SQD(\rho) = \overline{BQD(\rho)} \subset QD(\rho)$.

Definition 2.3 (Thurston's Compactification).

$\overline{\tau(S)}^{Th} = \tau(S) \cup PMF \subset$ projective length spectrum, where PMF denotes the projectivized measured foliation.

Theorem 2.4 (Wolf).

$\overline{\tau(S)}^W \cong \overline{\tau(S)}^{Th}$ (homeomorphism). In particular, $\phi \in SQD_\sigma \mapsto F_U(\phi)$.

Proof.

Given a family of hyperbolic metrics $\rho_t \to +\infty$ and a Riemann Surface Σ, a family of harmonic maps $u : (S, \sigma) \to (S, \rho_t) \sim$ id. such that $\phi(\rho_t) = t\phi$ fixed. By analyzing the elliptic equation relation ρ_t and $\phi(\rho_t)$, it turns out as $t \to +\infty, \rho_t$ is approximated by the transverse measure μ in the vertical measured foliation $F_u(t\phi)$.

3. Harmonic Maps to $\mathbb{R}$-trees

Consider leaves contracting on a surface. So we have

Definition 3.1.

$$(\widetilde{S}, \widetilde{\rho}_t) \overset{\mathrm{t}\to+\infty}{\longrightarrow} \text{leaf space of vertical measured foliations for } t\phi$$

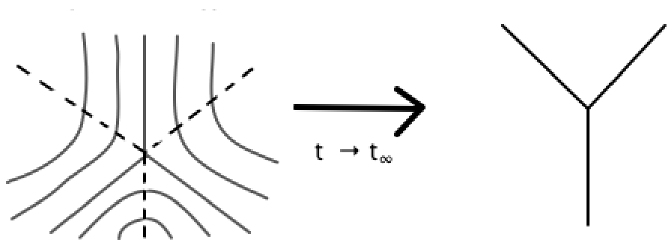

Figure 4: contracting leaves example

with metric $d = \rho_*(t^{\frac{1}{2}}\mu)$. Renormalizing it gives us

$$(\widetilde{S}, \frac{1}{t}\widetilde{\rho}_t) \overset{t \to +\infty}{\longrightarrow} \text{leaf space of vertical measured foliations for } F_u(\phi)$$

with metric $d = \rho_*(\mu)$.

We say that the leaf space is a $\mathbb{R}$-*tree* and denote it by T_ϕ.

So the limit of the harmonic maps id $\sim u : (S, \sigma) \to (S, \frac{1}{t^{\frac{1}{2}}}\rho_t)$ is $\rho : (\widetilde{S}, \sigma) \to (T_\phi, d)$ where $d = \pi_*(\mu)$ by projection to the leaf space. Away from zeros we note that ρ is locally harmonic. ρ is harmonic in general.

Conclusion: In SL$(2, \mathbb{R})$ case, the holomorphic quadratic differential q_2 can:

- guide the limit behavior of representations to $+\infty$ and harmonic maps
- gives a limit object $\mathbb{R}$-tree Tq_2 serves as target of limit harmonic maps (The limit harmonic map $\rho : \Sigma \to Tq_2$ by projecttion is for free.)

We may also generalize this case and ask about the $SL(n, \mathbb{C}), SL(n, \mathbb{R})$ cases. In fact, both these cases are conjectured in [KNPS].

For $SL(n, \mathbb{C})$ case, we ask if the holomorphic differentials $(q_2, ..., q_n)$ can

1. Predict limit behavior of certain families of representations $\to +\infty$? Are they enough?

2. Be associated to a limit object (building) that we construct which admits limit representation actions?

Results.

If we examine the above question 1, [Loftin, DW] give partial results for $SL(3, \mathbb{R})$, [C-L] gives even partial results for $SL(n, \mathbb{R})$, and [KNPS] give results for $SL(n, \mathbb{C})$

for some families that are not in our setting.

If we examine the above question 2 for $SL(2,\mathbb{R}), SL(2,\mathbb{C})$, the object that we construct [Farb-Wolf-DDW] are *Harmonic splittings.* Here, the leaf space T_ϕ has a "universal" property. That is, for any harmonic map $u : \widetilde{\Sigma} \to (T,d)$,

$$\mathrm{Hopf}(u) = \phi \text{ on } \Sigma.$$

Then there exists an equivariant tree-morphism: $(T,\phi) \to (T,d)$ such that

$$\begin{array}{ll} \tilde{\Sigma} \to T_\phi & \\ \quad\curvearrowright & \\ \quad\searrow \downarrow & \\ \quad(T,d) & \end{array}$$

preserving uniqueness.

In the $SL(n,\mathbb{C})$ case, we may consider a family of representations, $\rho_t \to +\infty$. This then gives a family of ρ_t-equivalent harmonic maps,

$$u_t : \tilde{\Sigma} \to SL(n,\mathbb{C})/SU(n).$$

Also, as $t \to \infty$,

$$u_\infty : \tilde{\Sigma} \to \text{"Building-like" object}$$

(Anne Parreau uses an asymptotic cone as a "building-like" object for compactification).

Finally, we have a conjecture summarizing the results that relate to question 2.

Conjecture (KNPS).

For any $\phi = (q_2, ..., q_n)$, we can associate a universal ϕ-building B^ϕ such that if u is harmonic map $u : \tilde{\Sigma} \to$ Building B associated to $\phi = (q_2, ..., q_n)$ holomorphic differentials, then there exists an equivalent building morphism:$B^\phi \to B$ such that

$$\begin{array}{l} \tilde{\Sigma} \to B^\phi \\ \\ \quad\searrow \downarrow \\ \quad B \end{array}$$

preserving uniqueness.