

# What We Knew About Hyperbolic Geometry Before We Knew About Hyperbolic Geometry

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We now discuss the history and milestones of how we came to know what we know about hyperbolic geometry. A Hyperbolic geometry scorecard will list some key advances made in chronological order.

## 1. The Greeks

### 1.1 Euclid's Parallel Postulate

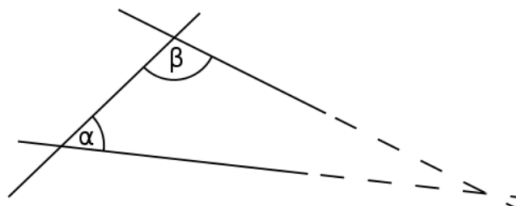


Figure 1: Euclid's Parallel Postulate

Euclid stated in his Parallel Postulate, “If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.”

REMARKS (WHY EUCLID'S PARALLEL POSTULATE IS UPSETTING).

(i) It's clearly true, but it takes a long time to say.

(ii) Proclus (5th century CE, as quoted by Heath): “This ought even to be struck out of the Postulates altogether; for it is a theorem involving many difficulties,...And the converse of it is actually proved by Euclid himself as a theorem.” (Book 1, proposition 17: the sum of two angles of a triangle is less than two right angles.)

(iii) New rule: If you are true, and your converse is a theorem, you should be provable.

### 1.2 The rest of Euclid's theory of parallels

The remaining statements in Euclid's theory of parallels are given by the following propositions:

PROPOSITIONS 27 + 28.

If a transversal to two straight lines makes the alternate angles equal to one another, or the interior angles equal to two right angles, then the straight lines are parallel to each other.

PROPOSITION 29.

A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

PROPOSITION 30.

Straight lines parallel to the same straight line are parallel to one another.

PROPOSITION 31 (PLAYFAIR'S AXIOM).

Through a given point one and only one straight line can be drawn which will be parallel to a given straight line.

### 1.3 Early Greeks

Euclid defines parallel with definition, "Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction."

We may also consider an alternative definition of parallel given by Posidonius (1st Century BCE). He stated that parallel lines are coplanar straight lines that are equidistant. Therefore, this definition leads to the Euclidean theory of parallels without need of a parallel postulate.

We now may add a fact to our Hyperbolic geometry scorecard:

*In non-Euclidean geometry, parallel lines are not equidistant (1st century BCE).*

What might be wrong with Posidonius' definition? The answer to the question is asymptotes! Are asymptotes straight lines? The definitions "a line is breathless length" and "a straight line is a line which lies evenly with the points on itself" aren't great at distinguishing our concept of usual straight lines from asymptotes. We can't just pretend that we know what straight lines are.

Proclus adds, "It may be that some would be deceived and would think it proper to place even the assumption in question among the postulates as affording, in the lessening of the two right angles, ground for an instantaneous belief that the straight lines converge and meet. To such as these Geminus correctly replied that we have

learned from the very pioneers of this science not to have any regard to mere plausible imaginings when it is a question of the reasonings to be included in our geometrical doctrine.”

### 1.4 Ptolemy (2nd century CE)

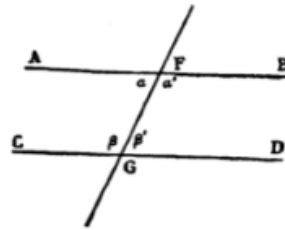


Figure 2: Ptolemy's argument

In Figure 2,  $\alpha + \beta$  must be greater than, less than, or equal to  $\alpha' + \beta'$ . (And hence greater than, less than, or equal to two right angles.) A very weird hidden assumption that is made is:

The symbol that holds for one pair of parallels holds for all.

### 1.4 Proclus

Proclus did not approve of Ptolemy's "proof" and had his own.

But before sharing that, we look at a different notion of parallel, the argument that Proclus criticizes.

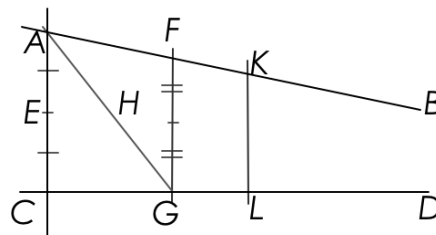


Figure 3: Proof that the lines in Euclid's postulate can't meet

We may follow this argument with the steps: (See Figure 3)

1. Interior angles formed by  $AG$  don't add up to 2 right angles.

2. Bisect  $AC$  with  $E$ .
3. We know that  $F$  and  $G$  are apart.
4. Bisect  $FG$  with  $H$ .
5. We note that the lengths of  $CG$  and  $AF$  are the same as the length of  $AE$  which allows us to say that  $CG$  does not meet  $AF$  at  $F$  (Since if  $G$  were the same length as  $F$  we would have a triangle in which the length of one side was equal to the sum of the lengths of the other two sides).
6. Likewise,  $GL$  and  $FK$  are the same length as  $FH$ , and the same argument holds.
7. Repeat this process forever and conclude that the two lines  $AE, CD$  never intersect.

In response to the proof above, Proclus gives his criticism that we can join  $A$  to  $G$  and therefore obtain lines that do intersect. So he believes the above proof proves too much. So he comments,

“Therefore it is not possible to assert, without some definite limitation, that the straight lines produced from angles less than two right angles do not meet. On the contrary, it is manifest that some straight lines, when produced from angles less than two right angles, do meet, although the argument seems to require it to be proved that this property belongs to all such straight lines. For one might say that, the lessening of the two right angles being subject to no limitation, *with such and such an amount of lessening the straight lines remain non-secant, but with an amount of lessening in excess of this they meet.*” (Heath stated the part that is emphasized.)

### 1.5 1400 years later...

Lobachevsky stated in *The Theory of Parallels*, “All straight lines which in a plane go out from a point can, with reference to a given straight line in the same plane, be divided into two classes—into cutting and non-cutting. The boundary lines of the one and the other class of those lines will be called *parallel* to the given line.”

So we can add another fact to our Hyperbolic geometry scorecard:

*Partial credit: There might be “extremal” parallel lines (5th century CE).*

### 1.6 Proclus’ proof of the parallel postulate

Proclus gives us two different types of assumptions:

1. Explicit assumption: The distance between two points upon two intersecting straight lines can be made as great as we please, by prolonging the two lines sufficiently.

2. Hidden assumption: The distance between two parallel lines is finite.

PROPOSITION (PROCLUS).

A straight line which meets one of two parallels must also meet the other.

PROOF.

Referring to Figure 4, we have that  $AB$  and  $CD$  are a finite distance apart. This implies that  $FEG$  must eventually cross  $CD$ . Since it already crosses  $AB$ , and with the postulates relating exterior and or interior angles of the figure, we have that  $AB$  and  $CD$  are parallel.

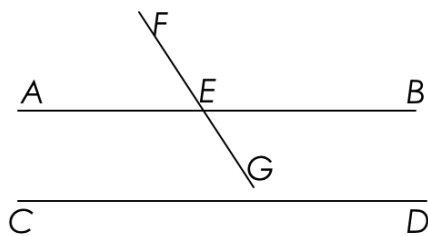


Figure 4: Proof giving  $AB$  parallel to  $CD$

THEOREM (PARALLEL POSTULATE).

Let  $AB, CD$  be two straight lines, and let  $EF$  falling on them make the angles  $BEF, DFE$  less than two right angles. I say that the straight lines will meet on that side on which are the angles less than two right angles.

PROOF.

Referring to Figure 5, we build  $HE$  such that  $GEF + DEF = 2$  right angles. So by Euclid's proposition 28,  $HG$  is parallel to  $CD$ .

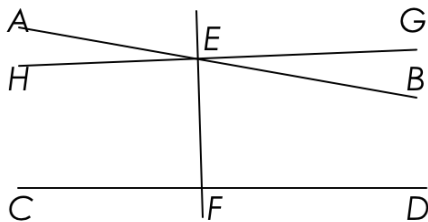


Figure 5: Proof of Parallel Postulate

Now we can add the following to our Hyperbolic geometry scorecard:  
*In non-Euclidean geometry, parallel lines get arbitrarily far from each other (5th century CE).*

## 2 Medieval Islamic mathematicians

### 2.1 al-Gauhary (9th century) and al-Haytham (965 - 1039)

al-Gauhary approaches this question by using the proposition:

Through any point interior to an angle, it is possible to draw a line that intersects both sides of the angle.

al-Gauhary makes a hidden assumption:

If alternating angles determined by a line cutting two other lines are equal, then this will hold for all lines cutting the given two.

In his proof by contradiction using motion and transformation, al-Haytham gave the first use of “Lambert quadrilaterals.”



Figure 6: Lambert quadrilateral

### 2.2 Omar Khayyam (1048-1131)

Omar Khayyam wrote *Commentary on the Difficulties of Certain Postulates of Euclid's Work* where he adds his own treatment of the parallel postulate and some of the later postulates about proportions. Khayyam did not try to prove the parallel postulate from Euclid's other postulates but thought that Euclid had picked the “wrong” postulates to start the book. He proposed adding five postulates and replacing the 29th proposition, the first to use the parallel postulate, with eight new propositions.

SIDE REMARK.

Omar Khayyam is almost universally identified with the author of the *Rubaiyat*, one of the most famous collections of Persian poetry. However, this might just be a deeply entrenched myth.

### 2.3 On the true nature of parallels and the exposition of the celebrated difficulty

Specific complaints from Khayyam's Commentary on Some Difficulties in the Postulates of Euclid include:

1. Euclid proved that equal angles at the centers of equal circles cut off equal arcs from the circumference and that the ratio of the same magnitude to two equal magnitudes is the same. On the other hand, in the sections on solids, he didn't prove as much material, "Do you not see that whosoever conceives the reality of the circle, the reality of the angle, and the reality of the ratio between magnitudes, will know with a modicum of reflection, that the ratio of the angles which stand at the center is equal to the ratio of the arcs on which they stand? Yet this notion has been demonstrated by Euclid in proposition 36 of Book VI."

2. "And there are more obvious primary premises than this; but Euclid has not produced most of them in the beginning of the work, although he has produced primary things which one could very well do without. And he had to either not produce them at all, or produce all of them without excluding any of them, even if they are obvious."

#### NEW POSTULATES.

Postulates that we have acquired from Khayyam (influenced by Aristotle) are given by:

- Magnitudes are divisible ad infinitum, and are not composed of indivisible things.
- One can produce a straight line to infinity.
- Whenever two straight lines intersect, they will diverge when going away from the angle of intersection.
- Two converging straight lines will intersect, and two converging lines cannot diverge while going toward convergence.
- Whenever there be two unequal finite magnitudes, the smaller can be multiplied until it becomes greater than the greater. (Also known as Archimedes' axiom.)

Omar Khayyam was the first to use what are now known as "(Khayyam-)Saccheri quadrilaterals" (650 years before Saccheri). These are quadrilaterals with a base and two congruent sides that are perpendicular to the base. The other side is called the "summit."

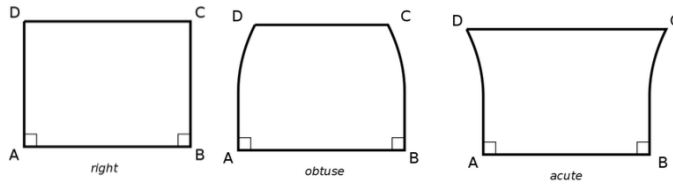


Figure 7: Khayyam-Saccheri quadrilaterals

PROPOSITION.

The line  $AB$  is given. And we draw  $AC$  perpendicularly to  $AB$ , and we set  $BD$  perpendicular to  $AB$  and equal to the line  $AC$  (they will consequently be parallel, as has been demonstrated by Euclid in Proposition 26 [28]), and we join  $CD$ . I say then that the angle  $ACD$  is equal to the angle  $BDC$ .

PROOF.

Similar triangles!

PROPOSITION.

We repeat the figure  $ABCD$ , and we divide  $AB$  in two in  $E$ , and we draw  $EG$  perpendicularly to  $AB$ . I say then that  $CG$  is equal to  $GD$ , and  $EG$  perpendicular to  $CD$ .

PROOF.

Similar triangles!

So we can add the following to our Hyperbolic geometry scorecard:

*In neutral geometry, the quadrilaterals we unjustly know as Saccheri quadrilaterals have congruent summit angles, and the perpendicular bisector to the base is a perpendicular bisector of the summit (11th century CE).*

PROPOSITION.

And we repeat the figure  $ABCD$ . I say then that the angles  $ACD, BDC$  are right.

PROOF.

Acute angles are too small. Obtuse angles are too large. Right angles are just right.

So we can add the following to the Hyperbolic geometry scorecard:

*In hyperbolic geometry, straight lines will diverge in both direction from their mutual perpendicular (11th century CE).*

Now we will find a definition that describes the distance between lines at a point



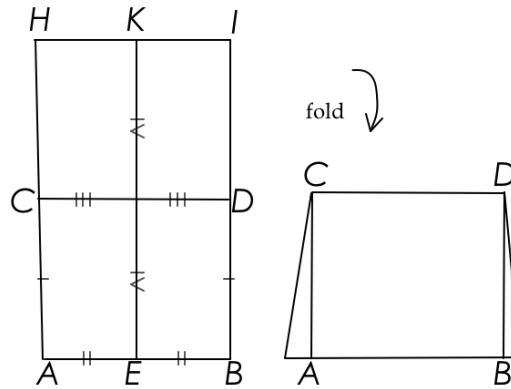


Figure 8: face-to-face

and what it means for lines to converge or diverge in a direction. Lines that we would call parallel are called “*face-to-face*.” This will allow us to reiterate the postulate, “Therefore it has become plain, that if two straight lines in a plane surface converge in one direction, it will absolutely be impossible that they diverge in that direction; and likewise if that diverge. Except that this demonstration is not a geometrical demonstration; it is only a philosophical demonstration.” For example, we look at Figure 8, with congruent edges as shown. Assume  $ACD$  is an acute angle. When we fold the left hand quadrilateral over  $CD$ , we may note that  $AC, BD$  diverge both “up” and “down;” they both diverge and converge and then diverge again. But this contradicts the fourth of the new postulates that Khayyam wanted to add to Euclid: two lines cannot converge and then diverge. This is a contradiction, so the lines must be face-to-face, and the summit angles must be right angles.

So far, we have the following propositions:

- $CD$  is congruent to  $AB$ .
- If two lines are face-to-face, then each line perpendicular to one of them will be perpendicular to the other.
- Parallel lines must be face-to-face.
- If a straight line falls upon two parallel lines, the alternate angles will be equal to one another, etc. (Euclid propositions 29 and 30.)
- Euclid’s parallel postulate.

Khayyam believes, “And the truth is, that one should add these propositions to the work the Elements according to the order which was mentioned;...For the art requires it in order to be brought to philosophical perfection, so that the one which looks into it will not be disturbed with doubts and uncertainties.”

## 2.4 Nasir al-Din al-Tusi (1201-1274)

Nasir al-Din al-Tusi studied the works of al-Haytham and Omar Khayyam. He attempted to prove parallel postulate by contradiction without using extra postulates.

## 2.5 John Wallis (1616-1703)

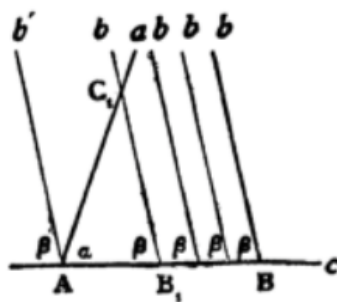


Figure 9: Wallis' Proof

Wallis proves that the lines given by  $a$  and  $b$  will have an intersection that forms a triangle. He has:

Assumption: For any triangle, you can make a similar triangle of any size.

Justification: Euclid postulates the existence of a circle of any radius. Why not triangles?

Referring to Figure 9, we can consider multiple parallel translates of the line  $b$  (on the right hand side of the picture). We can slide a line  $b'$  until it starts at  $a$  and at some point when it moved from the right to the left side of the picture it must have hit  $a$  in an intersection. In more detail we see that because Wallis postulates the existence of similar triangles of any size, we must be able to create a similar triangle with sides  $a$  and the original  $b$  with angles  $\alpha$  and  $\beta$ . Therefore  $a$  and  $b$  must intersect to create the third vertex of the triangle.

Furthermore, we can find an angle  $\gamma$  created by the intersection of  $a$  and  $b$  so there must be a triangle formed as well.

## 2.6 Giovanni Gerolamo Saccheri (1667-1733)

Euclid Freed of Every Flaw Reappearance of Khayyam-Saccheri quadrilaterals.

THEOREMS.

- If the hypothesis of the acute/right/obtuse angle holds, the top is longer/same/shorter than the bottom.
- If the hypothesis of the right, acute, or obtuse angle holds in one case, it holds in every case.
- For the hypothesis of the right/acute/obtuse angle, the angles in a triangle are equal to/less than/greater than two right angles.
- The hypothesis of the obtuse angle is right out. (Euclid proposition 17.) “The hypothesis of the acute angle is absolutely false, because it is repugnant to the nature of the straight line.”

## 2.7 Johann Heinrich Lambert (1728-1777)

Like al-Haytham, Johann Heinrich Lambert used motion and Lambert quadrilaterals. He made an interesting observation that perhaps hyperbolic geometry was like spherical geometry on a sphere of imaginary radius. We note that hyperbolic geometry isn't exactly geometry on a sphere of imaginary radius, but a related concept called the pseudosphere, the analog of the sphere that has constant negative rather than constant positive curvature.



Figure 10: A pseudosphere

## 2.8 Recommended Reading

1. The Elements of Euclid translated and with a commentary by Thomas Heath. (Free on archive.org and probably other places.)
2. Non-Euclidean Geometry, Roberto Bonola
3. Omar Khayyam the Mathematician, R. Rashed and B. Vahabzabeh
4. Why did Lagrange prove the parallel postulate? J.V. Grabiner

## 2.9 Alternate timeline questions

What if Euclid had used equidistance as the definition of parallel? What if Euclid had used an equivalent statement such as “You can make a triangle as big as you want to” instead of the parallel postulate?

### Hyperbolic Geometry scorecard

- In non-Euclidean geometry, parallel lines are not equidistant (1st century BCE)
- Partial credit: There might be extremal parallel lines (5th century CE)
- In non-Euclidean geometry, parallel lines get arbitrarily far from each other (5th century CE)
- In neutral geometry, the quadrilaterals we unjustly know as Saccheri quadrilaterals have congruent summit angles, and the perpendicular bisector to the base is a perpendicular bisector of the summit (11th century CE)
- In hyperbolic geometry, straight lines will diverge in both direction from their mutual perpendicular (11th century CE)
- In hyperbolic geometry, angles determine size. (17th century CE)
- In hyperbolic geometry, the internal angles of a triangle are less than two right angles. (18th century CE)