THE WEYL CHAMBER FLOW AND APPLICATIONS

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1. Rank 1

1.1. Basic example: PSL$(2,\mathbb{R})$. Consider the $\mathrm{PSL}(2,\mathbb{R})$ action on the upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ by $z \mapsto \frac{az+b}{cz+d}$. This action has several properties:

(1) It preserves the metric $\frac{4\mathrm{d}z\mathrm{d}\bar{z}}{(\Im(z))^2}$
(2) It is transitive, i.e. for all $q, z \in \mathbb{H}^2$, there exists $A \in \mathrm{PSL}(2,\mathbb{R})$ such that $Aq = z$.
(3) $\mathrm{Stab}(i) = \mathrm{SO}(2)$

We thus have that $\mathrm{PSL}(2,\mathbb{R})$ is an $\mathrm{SO}(2)$-principal bundle over $\mathbb{H}^2$.

There are three one-parameter subgroups of $\mathrm{PSL}(2,\mathbb{R})$ that we are interested in:

- $A = \left\{ \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \mid t \in \mathbb{R} \right\}$ - The right action of A on $\mathrm{SL}(2,\mathbb{R})$ preserves Haar measure λ.
- $N^+ = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}$ and $N^- = \left\{ \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}$

Note that

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} = \begin{pmatrix} 1 & se^{2t} \\ 0 & 1 \end{pmatrix}$$

so N^+ is expanded by A, in a similar manner one can see that N^- is contracted by A.

1.2. The rank-1 real Lie groups.

G	$\mathrm{SO}(n,1)$	$\mathrm{SU}(n,1)$	$\mathrm{Sp}(n,1)$	F_4^{-20}
K	$\mathrm{SO}(n)$	$S(U(n) \times U(1))$	$\mathrm{Sp}(n) \times \mathrm{Sp}(1)$	$\mathrm{Spin}(9)$

In each case, K is the maximal compact subgroup, G/K is a symmetric space of rank 1, and G is a K-principal bundle over G/K. We would like to view this as the *bundle of frames* over this symmetric space. There is a K-invariant Riemannian metric on G/K which projects from the K-invariant metric on G, this gives the symmetric space the structure of a Riemannian manifold. G is just the principal bundle of orthonormal frames in the case $G = \mathrm{SO}(n,1)$, in the case $G = \mathrm{SU}(n,1)$, where we have a complex structure, you take *unitary frames*.

Definition 1. A *unitary frame* consists of a basis over the tangent space to the complex numbers which is orthogonal to the hermitian form.

The unit tangent bundle to $\tilde{M} = G/K$, $T^1(\tilde{M}) = G/K_0$ is a homogeneous space. Where K_0, in the case of $G = \mathrm{SO}(n-1)$, is $\mathrm{SO}(n-1) = \mathrm{Stab}_{\mathrm{SO}(n)}(e \in S^{n-1})$.

Fact 1. *$PSL(2,\mathbb{R}) = SO(2,1) \hookrightarrow SO(n,1)$ by $(x_0,\dots,x_n)\cdot(y_0,\dots,y_n) = -x_0y_0 + x_1y_1 + \cdots + x_ny_n$. In particular, this implies that the subgroup $A \subset PSL(2,\mathbb{R})$ embeds.*

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1.3. **Lattices, the geodesic flow and the frame flow.** Let $\Gamma \subset G$ be a cocompact lattice. Then $\Gamma\backslash G/K_0 = T^1M$, where $M = \Gamma\backslash G/K_0$. We define the following flows:

- *geodesic flow* Φ^t: A vector $v \in T^1M$ defines a geodesic in M that is a solution to an ordinary differential equation with initial velocity v. Then set $\Phi^t v = \gamma_{v'}(t)$.
- *frame flow* Ψ^t: $(v_1, \ldots, v_n) \mapsto (\Phi^t v_1, ||v_2, \ldots, ||v_n)$, where $||v_i$ indicates parallel transport. This flow moves and orthonormal (resp. unitary) fram to an orthonormal (resp. unitary) frame.

Goal: Understand what one can do with this frame flow.

Observation 1. We get a probability measure λ on $\Gamma\backslash G$, as the normalized projection of Haar measure, which is invariant under the frame flow.

We know that $L_0^2(\Gamma\backslash G) = \{f : \Gamma\backslash G \to \mathbb{R} | \int f^2 \mathrm{d}\lambda < \infty, \int f \mathrm{d}\lambda = 0\}$ which gives us a representation of G on $L_0^2(\Gamma\backslash G) = \mathcal{H}$ where $gf(h) = f(hg)$.

Definition 2. A vector is v is *K-finite* if the dimension of Kv, the span of all translates of v, is finite.

Definition 3. (Harish-Chandra) A unitary representation ρ of G is *tempered* if for all K-finite unit vectors $v, w \in \mathcal{H}$

$$|\langle \rho(g)v, w\rangle| \leq ((\dim Kv)(\dim Kw))^{1/2}\Theta_\rho(g)$$

where Θ_ρ is the Harish-Chondra function.

Theorem 1 (Cowling-Haagrup-Howe)**.** *Fix $k \geq 2$. Then, for $f_1, f_2 \in C^k(\Gamma\backslash G)$, there is a constant $\kappa = \kappa(\Gamma)$ such that*

$$\left|\int (f_1 \circ \Psi^t) f_2 - \left(\int f_1\right)\left(\int f_2\right)\right| \leq e^{-\kappa t}||f_1||_{C^k}||f_2||_{C^k}$$

Remark 1. The frame flow is defined for any closed Riemannian manifold.

If the curvature is negative, we have the following:

- Dolgopyat, Liverani: The geodesic flow is exponentially mixing which implies that exponential decay of correlation holds.
- Brin-Gromov: The frame flow is mixing for odd-dimensional manifolds

Open problem: What can you say about the rate of mixing for "generic" metrics?

2. Applications to results on the structure of lattices

Theorem 2 (Hamenstädt; Kahn-Markovich for $G = \mathrm{SO}(3,1)$)**.** *Consider $G \neq SO(2m, 1)$ for $m \geq 2$. Let $\Gamma < G$ be a cocompact lattice. Then Γ contains a surface subgroup, that is, a fundamental group of a closed surface of genus $g \geq 2$.*

Remark 2. Two vectors v, w span a totally real subspace if $w \perp \{v, Jv\}$, where J is the complex structure. A totally real orthonormal 2-frame is tangent to a totally geodesic hyperbolic plane.

Proof sketch: Any *tripod*, i.e. three tangent vectors at angles of $\frac{2\pi}{3}$, defines an ideal triangle. We first construct a pair of pants (thrice punctured sphere), by gluing two truncated ideal triangles. Mixing allows us to connect v_1 and w_1. W can take a vector in a small neighborhood of v_1 and it will connect to a vector in a small neighborhood around w_1. Doing this with all three ends allows us to construct a closed curve γ that is homotopic to a closed geodesic (See, Fig. 1). Since we can control the length of the boundary of the pair of pant, we can uniformly control the lengths of these closed geodesics.

Gluing the pairs of pants with the same boundary produces a closed surface $S \to \Gamma\backslash G/K$ and thus an embedding $\pi_1 S \to \Gamma$. This gives you criterion expressed in the form of a gluing equation which can be solved using exponential mixing. □

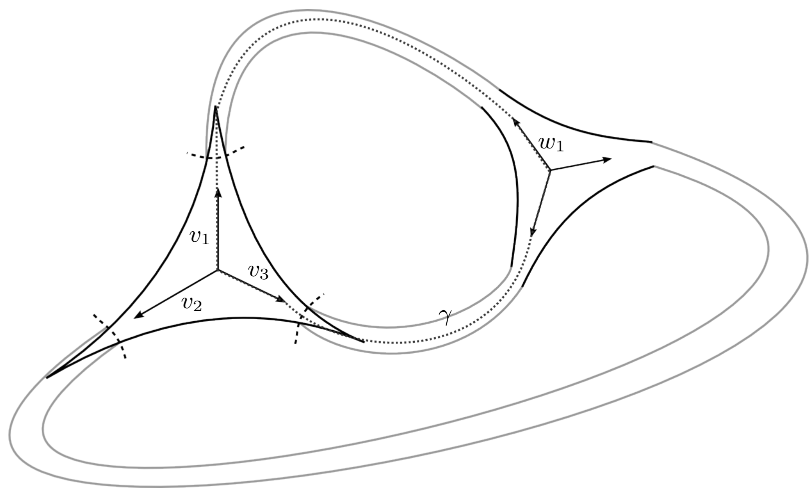

FIGURE 1. Finding a closed curve between two tripods