Some number-theoretic tools used in homogenous dynamics

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LECTURE 1. Primer on number fields

Definition

$\alpha \in \mathbb{C}$ is an algebraic number if there is $f \in \mathbb{Z}[X]$ not identically 0 s.t. $f(\alpha) = 0$. α is an algebraic integer if f can be chosen to be monic.

Definition

A number field K is a finite extension of $\mathbb{Q}$: $[K : \mathbb{Q}] = n \geq 2$; n is the degree of K.

Theorem

Let K be a number field of degree n, then

- $\exists\ \theta \in K$ such that $K = \mathbb{Q}(\theta)$ (primitive element). Its minimal polynomial $f \in \mathbb{Q}[X]$ is an irreducible polynomial of degree n.
- $\exists$ exactly n field embeddings $\varphi_i : K \to \mathbb{C}$, $\theta \mapsto \theta_i$, (θ_i are roots of f in $\mathbb{C}$). φ_i are $\mathbb{Q}$-linear, $K_i = \varphi_i(K)$ are isomorphic to K.
- Elements of K_i are algebraic numbers: $1, a, \ldots, a^{n-1}$ are lin. dep.

Primer on number fields

$$K = \mathbb{Q}[X]/(f(X))$$

Examples: (1) $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$. (2) $\mathbb{Q}[X]/(X^3 - 2) = \mathbb{Q}(\theta)$ has one real and two complex conjugate embeddings:
$\theta_1 \mapsto {}^3\sqrt{2}, \theta_2 = {}^3\sqrt{2}\omega, \theta_3 = -{}^3\sqrt{2}\omega$, where $\omega^3 = 1$.

Definitions

(1) The set of algebraic integers in K form a ring $\mathcal{O}_K$ called the ring of integers in K, it is a maximal order (a subring of K which as $\mathbb{Z}$-module is finitely generated) of rank n. $\mathbb{Z}[\theta] \subset \mathcal{O}_K$ of finite index.
(2) A $\mathbb{Z}$-basis of $\mathcal{O}_K$ is called an integral basis of K.
(3) $\mathrm{Tr}\,(\alpha) = \sum_{i=1}^n \varphi_i(\alpha)$ - trace, $N(\alpha) = \prod_{i=1}^n \varphi_i(\alpha)$ - norm.
(4) $\alpha \in \mathcal{O}_K$ is called a unit if $1/\alpha \in \mathcal{O}_K$, equivalently, if $N(\alpha) = \pm 1$.
(5) Let $\alpha_1, \dots, \alpha_n$ be an integral basis of K. Then $\det(\mathrm{Tr}\,(\alpha_i \alpha_j)) := d(K)$, the discriminant of K. It does not depend on the choice of an integral basis in K.

Geometric representation of algebraic numbers

Definition

The signature of a number field K is (r_1, r_2) where r_1 is the number of real and r_2 is the number of pairs of non-real complex embeddings.

Rather than combining all these embeddings into a single embedding of K into $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ (which is non-canonically $\mathbb{R}^n$, even if K is totally real, since it depends on the choice of the ordering of your embeddings), work instead with $K_\mathbb{R} = K \otimes_\mathbb{Q} \mathbb{R}$ that has a natural $\mathbb{R}$-algebra structure. The field isomorphism $K \to \mathbb{Q}[X]/(f(X))$ extends naturally to an algebra isomorphism $K_\mathbb{R} \to \mathbb{R}[X]/(f(X))$ and since $f = \prod_{i=1}^{r_1}(X - x_i) \times \prod_{j=1}^{r_2}(X^2 - t_j X + n_j)$ we obtain an isomorphism of $\mathbb{R}$-algebras:

$$K_\mathbb{R} \cong \mathbb{R}[X]/(f(X)) \cong \prod_{i=1}^{r_1} \mathbb{R}[X]/((X - x_i)) \times \prod_{j=1}^{r_2} \mathbb{R}[X]/((X^2 - t_j X + n_j))$$
$$\cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \cong \mathbb{R}^n.$$

Lattices and algebraic integers

Let $\{\alpha_1, \dots, \alpha_n\}$ be a $\mathbb{Z}$-basis of $\mathcal{O}_K$. The image of $\mathcal{O}_K$ is a lattice Λ in $\mathbb{R}^n$ with $\mathbb{Z}$-basis $\{\varphi_1(\alpha_j), \dots, \varphi_n(\alpha_j)\}$, i.e. a discrete additive subgroup of $\mathbb{R}^n$, or, equivalently, a finitely generated free $\mathbb{Z}$-module with positive definite symmetric bilinear form $(\alpha_i, \alpha_j) = \mathrm{Tr}\,(\alpha_i \alpha_j)$, called the trace product (pairing). Let $G = \mathrm{Tr}\,(\alpha_i \alpha_j)$. Let $G = AA^T$, where $A = (\varphi_i(\alpha_j))$. Since $vol(\mathbb{R}^n/\Lambda) = |\det(A)|$, and discriminant of Λ, $d(\Lambda) = \det(G)$, we have $d(K) = d(\Lambda) = vol(\mathbb{R}^n/\Lambda)^2$.

Since $vol(\mathbb{R}^n/\Lambda)$ were defined using complex embeddings of α_i: $(\sigma_1(\alpha_i), \dots, \sigma_{r_1}(\alpha_i), \tau_1(\alpha_i), \bar{\tau}_1(\alpha_i), \dots)$ (matrix A) while $vol(K_{\mathbb{R}}/\Lambda)$ is computed using $(\sigma_1(\alpha_i), \dots, \sigma_{r_1}(\alpha_i), \mathrm{Re}\,\tau_1(\alpha_i), \mathrm{Im}\,\tau_1(\alpha_i), \dots)$ (matrix B). Since $\det(A) = (-2i)^{r_2}\det(B)$, we have

Theorem

Under the geometric representation of algebraic numbers of the field K by the points of $\mathbb{R}^n$, all points representing the ring of integers $\mathcal{O}_K$ of discriminant $d(K)$ form a full lattice s.t. $d(K) = (-4^{r_2})vol(K_{\mathbb{R}}/\Lambda)^2$.

Geometric representation of algebraic numbers

Examples:

1. $K = \mathbb{Q}(\sqrt{-1})$. $\mathcal{O}_K = \mathbb{Z}[i]$, $d(K) = \det(A)^2 = \det \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^2 = -4$,
   $K_{\mathbb{R}} \cong \mathbb{C}$ by $a + b\sqrt{-1} \mapsto (a, bi) \in \mathbb{C}$,
   $vol(\mathbb{C}/\mathcal{O}_K) = \det(B) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$.
2. $F = \mathbb{Q}(\sqrt{-3})$. $\mathcal{O}_F = \mathbb{Z}[\omega]$, where $\omega = e^{2\pi i/3}$.
   $d(K) = \det(A)^2 = \det \begin{pmatrix} 1 & \omega \\ 1 & \bar{\omega} \end{pmatrix}^2 = -3$,
   $F_{\mathbb{R}} \cong \mathbb{C}$ by $a + b\sqrt{-3} \mapsto (a, \sqrt{3}bi) \in \mathbb{C}$,
   $vol(\mathbb{C}/\mathcal{O}_F) = \det(B) = \det \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{\sqrt{3}}{2}$.
3. $E = \mathbb{Q}(\sqrt{5})$. $\mathcal{O}_E = \mathbb{Z}[\gamma]$, where $\gamma = \frac{1+\sqrt{5}}{2}$.
   $d(E) = \det(A)^2 = \det \begin{pmatrix} 1 & \gamma \\ 1 & \gamma' \end{pmatrix}^2 = 5$,
   $E_{\mathbb{R}} \cong \mathbb{R}^2$ by $a + b\sqrt{5} = (a + \sqrt{5}b, a - \sqrt{5}b) \in \mathbb{R}^2$,
   $vol(\mathbb{R}^2/\mathcal{O}_E) = |\det(B)| = \left|\det \begin{pmatrix} 1 & 1 \\ -\frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{pmatrix}\right| = \sqrt{5}$.

The Dirichlet Unit Theorem

The set of units in K form a multiplicative group denoted by U_K.

Theorem (Dirichlet)

Let (r_1, r_2) be the signature of K. Then U_K is a finitely generated group of rank $r_1 + r_2 - 1$, i.e. $U_K \approx F \times \mathbb{Z}^{r_1+r_2-1}$, where F is a finite cyclic group - the group of roots of unity in U_K.

The main tool in the proof is the logarithmic embedding $L : K^* \to \mathbb{R}^{r_1+r_2}$ given by $x \mapsto (L_1(x), \dots, L_{r_1+r_2}(x))$, where $L_i(x) = \log|\varphi_i(x)|$ for $1 \le i \le r_1$, and $L_i(x) = \log|\varphi_i(x)|^2$ for $r_1 + 1 \le i \le r_2$.

- $L(U_K)$ is a lattice of rank $r = r_1 + r_2 - 1$ in the hyperplane $\sum_{i=1}^{r_1+r_2} x_i = 0$ of $\mathbb{R}^{r_1+r_2}$.
- The kernel of L is the group of roots of unity F.

The Regulator

The units $\varepsilon_1, \dots, \varepsilon_r$ whose existence was established by Dirichlet Theorem are called the fundamental units in K. A system of units $\varepsilon_1, \dots, \varepsilon_r$ is fundamental iff $L(\varepsilon_1), \dots, L(\varepsilon_r)$ form a basis in the lattice $L(U_K)$. The volume of the lattice $L(U_K)$ can be computed by adding to this system a vector $(\frac{1}{\sqrt{r+1}}, \dots, \frac{1}{\sqrt{r+1}})$ perpendicular to the hyperplane of length 1, and computing the the $r+1$-dimensional volume V. It is equal to the absolute value of the determinant of the matrix

$$\begin{pmatrix} L_1(\varepsilon_1) & L_2(\varepsilon_1) & \cdots & L_{r+1}(\varepsilon_1) \\ \cdots & \cdots & \cdots & \cdots \\ L_1(\varepsilon_r) & L_2(\varepsilon_r) & \cdots & L_{r+1}(\varepsilon_r) \\ \frac{1}{\sqrt{r+1}} & \frac{1}{\sqrt{r+1}} & \cdots & \frac{1}{\sqrt{r+1}} \end{pmatrix},$$

and we have $V = \sqrt{r+1} R_K$, where R_K is the absolute value of one of the minors of order r of the above matrix (sum of each row $= 0$) R_K is called the regulator of K.

Actions by commuting total automorphisms

Let $A \in SL(n, \mathbb{Z})$ with an irreducible characteristic polynomial f and hence distinct eigenvalues. Consider the action α on $\mathbb{T}^n$ by matrices commuting with A.

The centralizer of A in $M(n, \mathbb{Q}) \approx \mathbb{Q}[X]/(f(X)) = K = \mathbb{Q}(\lambda)$, where λ is an eigenvalue of A, by the map $\gamma : p(A) \mapsto p(\lambda)$ with $p \in \mathbb{Q}[X]$.

Let $v = (v_1, \dots, v_n)$ be an eigenvector of A with eigenvalue λ, its coordinates belong to K. If $B = p(A)$ then $Bv = p(\lambda)v = \mu v$, i.e. $\gamma(B) = \mu$. By taking different embeddings of this relation we obtain $B\varphi_j(v) = \varphi_j(\mu)\varphi_j(v)$, i.e. B is diagonalizable over $\mathbb{R}$ simultaneously with A.
If $B \in M(n, \mathbb{Z})$, its minimal polynomial is monic with integer coefficients, hence $\gamma(B) = \mu$ is an algebraic integer.
if $B \in GL(n, \mathbb{Z})$ then $N(\gamma(B)) = \pm 1$, i.e. $\gamma(B) = \mu$ is an algebraic unit.

Centralizers and algebraic number fields

Therefore,

- $C(A)$ = centralizer of A in $M(n,\mathbb{Z}) \approx$ an order $\gamma(C(A)) \subset \mathcal{O}_K$,
- $Z(A)$ = centralizer of A in $GL(n,\mathbb{Z}) \approx$ the group of units in $\gamma(C(A)) \Rightarrow$, by the Dirichlet Unit Theorem, to $\mathbb{Z}^{r_1+r_2-1} \times F$.
- The action we are interested in is $\alpha \approx Z(A)$.
- If rank of $Z(A) = r_1 + r_2 - 1 = n - 1$, i.e. the action is Cartan $\Rightarrow$ $r_2 = 0$, i.e. K is a totally real number field, and $F = \{\pm 1\}$.

Let $A_1 = A, A_2, \ldots, A_{n-1} \in Z(A) \Leftrightarrow \varepsilon_1, \ldots, \varepsilon_{n-1} \in U_K$, multiplicatively independent units in K, and we have

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Proposition

A_i are simultaneously diagonalizable over $\mathbb{R}$ and conjugate to

$$\begin{pmatrix} \varepsilon_i & 0 & \cdots 0 & \\ 0 & \varphi_2(\varepsilon_i) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varphi_n(\varepsilon_i) \end{pmatrix}.$$

Mahler measure and entropy

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In many cases the entropy can be computed explicitly. For example, let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be an n-dimensional torus, and $T : \mathbb{T}^n \to \mathbb{T}^n$ a toral automorphism given by a matrix $A_T \in GL(n, \mathbb{Z})$ with eigenvalues $\lambda_1, \dots, \lambda_n$, that preserves the Lebesgue measure μ on $\mathbb{T}^n$. Then $h_\mu(T) = \sum_{|\lambda_i|>1} \log|\lambda_i|$.

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Definition

Let $P = a_0 z^n + a_1 z^{n-1} + \cdots a_n = a_0 \prod_{i=1}^{n} (z - \alpha_i)$.
The Mahler measure of P is defined to be $M(P) = |a_0| \prod_{|\alpha_i|\geq 1} |\alpha_i|$.

For the characteristic polynomial P of A_T, $a_0 = \pm 1$, and $\alpha_i = \lambda_i$.
Hence $M(P) = \prod_{|\lambda_i|\geq 1} |\lambda_i|$, and thus $h_\mu(T) = \log M(P)$.

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Theorem (Schintzel)

If P is a totally real polynomial of degree n, then $M(P) \geq \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{n}{2}}$.

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Lower bound for the entropy

Let $T : \mathbb{T}^n \to \mathbb{T}^n$ be an Anosov automorphism with real eigenvalues. Then $h_\mu(T) \geq nc$, where $c = \frac{1}{2}\log\left(\frac{1+\sqrt{5}}{2}\right)$. The absolute minimum $\log\frac{1+\sqrt{5}}{2}$ is achieved for $n = 2$ and the matrix $A_T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ with characteristic polynomial $x^2 - x - 1$ of discriminant 5.

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The reason is very simple: entropy of a group action measures the exponential growth of the number of distinguishable orbit segments against the volume of a ball in the word-lengh metric or (some left-invariant metric in the acting group) while for the smooth (or Lipschitz) actions this number growth no faster than exponentially against the radius of that ball. Thus for rank $k > 1$ the entropy is $\lim_{r\to\infty} \frac{\log(c^r)}{r^k} = \lim_{r\to\infty} \frac{\log c}{r^{k-1}} = 0$.

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- coincides with the usual notion for $k = 1$,
- takes non-zero values if entropy of all elements (of the suspension) of the action is positive, and
- is finite for smooth actions on compact manifolds with respect to any Borel probability invariant measure.

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D. Fried [DF] suggested a notion of entropy for smooth actions of higher rank abelian groups based on averaging approach that satisfies these properties - we call it the Fried average entropy in a joint paper with A. Katok and F. Rodriguez Hertz [KKRH]. It is defined via the entropy function.

[DF] D. Fried, *Entropy for smooth abelian actions*, Proc. of the Amer. Math. Society, **87** (1983), no. 1, 111–116.
[KKRH] A. Katok, S. Katok, F. Rodriguez Hertz, *The Fried average entropy and slow entropy for actions of higher rank abelian groups*, Geometric and Functional Analysis, **24** (2014), 1204 – 1228.

The entropy function

Let $G = \mathbb{Z}^a \times \mathbb{R}^b,\ a + b = k$, i.e. an abelian group of rank k.

Definition

The entropy function for α, denoted by h_μ^α, associates to $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$ the value of measure-theoretic entropy of $\alpha(\mathbf{t})$, i.e. $h_\mu^\alpha(\mathbf{t}) = h_\mu(\alpha(\mathbf{t}))$.

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Generally, this function is not known to possess any nice properties other than the obvious positive homogeneity of degree one and central symmetry. However, due to Huyi Hu [H], for smooth actions it is

1. convex;
2. it (or its natural extension to $\mathbb{R}^k$) is piece-wise linear:
3. if $h_\mu^\alpha(\mathbf{t})) > 0$ for all $\mathbf{t} \neq 0$, it defines a norm in $\mathbb{R}^k$;
4. if for some $\mathbf{t} \neq 0$ $h_\mu^\alpha(\mathbf{t})) = 0$, it defines a semi-norm in $\mathbb{R}^k$.

[H] H. Hu, *Some ergodic properties of commuting diffeomorphisms*, Erg. Th. and Dynam. Syst. **13** (1993), 73–100

Definition of the Fried average entropy

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The Fried average entropy h^*_α of an $\mathbb{R}^k$ action α is the inverse of the volume of the unit ball in the entropy function norm/semi-norm $B(h^\alpha_\mu))$ multiplied by the volume of the generalized octahedron, $\mathcal{O} = \{(x_i) \in \mathbb{R}^k | \sum_{i=1}^k |x_i| \leq 1\}$, equal to $\frac{2^k}{k!}$, i.e. $h^*_\alpha = \frac{2^k}{k! vol(B(h^\alpha_\mu))}$.

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- The Fried entropy h^*_α of a $\mathbb{Z}^k$ action α is defined as the Fried entropy of its suspension.
- For a mixed group $\mathbb{Z}^a \times \mathbb{R}^b$ with a $a + b = k$ one takes suspensions for the discrete generators.

The entropy function and Lyapunov exponents

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Lyapunov exponents of α for the invariant measure μ are linear functionals on $\mathbb{R}^k$, $\chi_j,\ 1 \leq j \leq n$, and their kernels are called the Lyapunov hyperplanes. Connected components of the complement to the union of the Lyapunov hyperplanes are called Weyl chambers.

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If μ is absolutely continuous, the entropy function is given by the Pesin entropy formula $h_\mu^\alpha(\mathbf{t}) = \sum_{j:\chi_j>0} \chi_j(\mathbf{t}),$ for $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$.

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In this case $\sum_{j=1}^n \chi_j = 0$, hence

- $h_\mu^\alpha(\mathbf{t}) = \frac{1}{2} \sum_{j=1}^n |\chi_j(\mathbf{t})|$.
- $h_\mu^\alpha(\mathbf{t})$ is linear inside each Weyl chamber.

Lyapunov exponents for the Cartan action

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For $n \geq 3$, let α be a Cartan action given by commuting matrices $A_i,\ 1 \leq i \leq n-1$, with real eigenvalues $\lambda_j(A_i), 1 \leq j \leq n$.
We define n Lyapunov hyperplanes on $\mathbb{R}^{n-1}$ by

$$\chi_j(\mathbf{t}) = \chi_j(t_1, \dots t_{n-1}) = \sum_{i=1}^{n-1} t_i \log|\lambda_j(A_i)| = 0 \tag{1}$$

for $1 \leq j \leq n$. In this case all hyperplanes are distinct. Since

$$\sum_{j=1}^{n} \log|\lambda_j(A_i)| = 0, \tag{2}$$

we have

$$\chi_1 + \chi_2 + \cdots \chi_n = 0,$$

hence inside each of $2^n - 2$ Weyl chamber the signs of χ_j's cannot be all the same.

The Weyl chambers and the entropy norm

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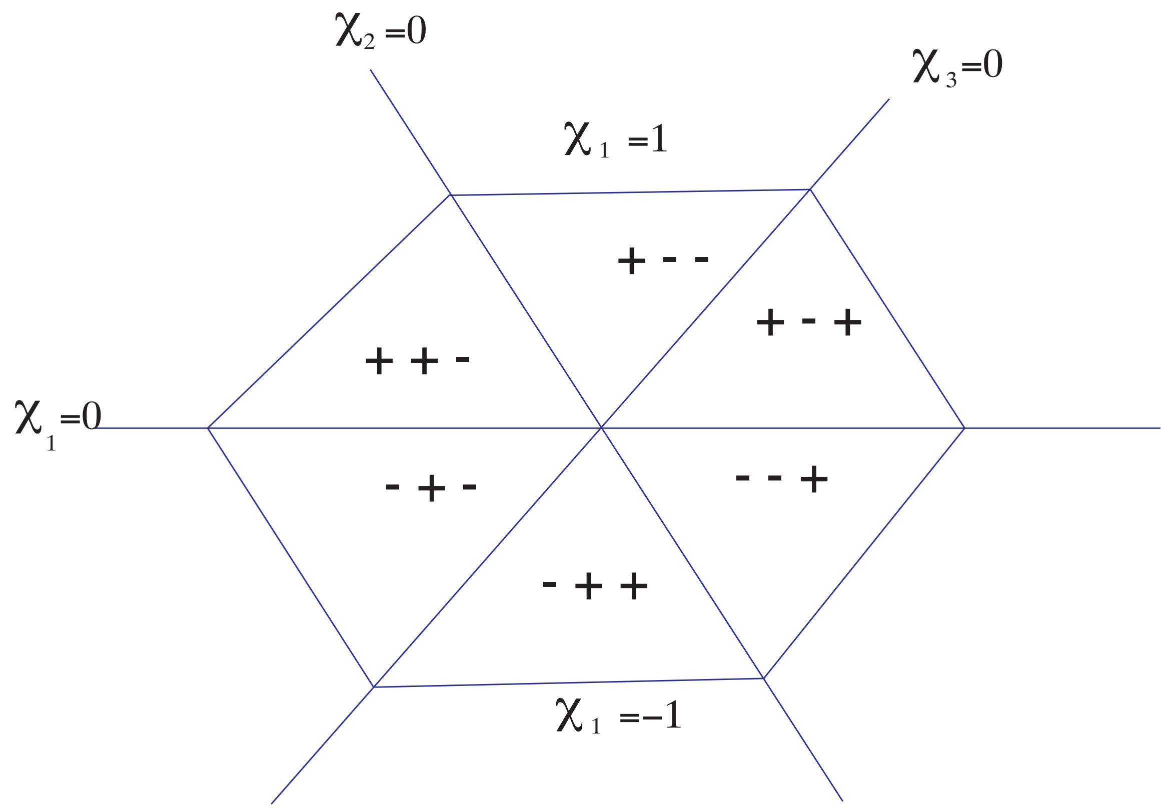

The Fried average entropy for maximal rank actions by total automorphisms

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Main Theorem [KKRH]

The Fried entropy of a maximal rank ergodic action by automorphisms of $\mathbb{T}^n$ of a given rank $n-1$ is greater than 0.089; furthermore the lower bound grows with n exponentially.

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- The formula for the Fried entropy for algebraic actions involved the regulator: $h_\alpha^* = \frac{kR_K 2^{n-1}}{\binom{2n-2}{n-1}}$, where R_K is the regulator of the field K and $k = [U_K : \gamma(Z(A))]$ is the index of $\gamma(Z(A))$ in the group of units U_K of the field K.

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[KKRH] A. Katok, S. Katok, F. Rodriguez Hertz, *The Fried average entropy and slow entropy for actions of higher rank abelian groups*, Geometric and Functional Analysis, **24** (2014), 1204 – 1228.
[K-RH] A. Katok and F. Rodriguez Hertz, *Arithmeticity and topology of smooth actions of higher rank abelian groups*, preprint.

Lower bounds for the Fried entropy

The lower bounds for the regulators will be found when $\gamma(Z(A)) = U_K$.
There are some lower bounds obtained by geometric methods using geometry of numbers.
We use Zimmert's analytic lower bound for regulators [Z]: for a totally real number field $[K : \mathbb{Q}] = n$, and any $s > 0$

$$R_K \geq a(s)\exp(b(s)n), \tag{3}$$

where

$$a(s) = (1+s)(1+2s)\exp\left(\frac{2}{s} + \frac{1}{1+s}\right)$$

and

$$b(s) = \log\left(\frac{\Gamma(1+s)}{2}\right) - (1+s)\frac{\Gamma'}{\Gamma}\left(\frac{1+s}{2}\right).$$

We need $b(s) > \log 2$, and for $s = 0.35$ we obtain

$$R_K > 0.000376\exp(0.9371n). \tag{4}$$

[Z] R. Zimmert, *Ideale kleiner Norm in Idealklassen and eine Regulatorabschätzung*, Invent. Math. **62** (1981), 367–380.

A lower bound for the Fried entropy

Using the estimate for the middle binomial coefficient

$$\frac{4^{n-1}}{n} \leq \binom{2n-2}{n-1} \leq 4^{n-1} \tag{5}$$

we obtain

$$\frac{1}{2^{n-1}} \leq \frac{2^{n-1}}{\binom{2n-2}{n-1}} \leq \frac{n}{2^{n-1}}.$$

Using the bound (4), we obtain

$$h^*_\alpha = \frac{R_K 2^{n-1}}{\binom{2n-2}{n-1}} > \frac{0.000376\exp(0.9371n)}{2^{n-1}} > 0.000752\exp(0.244n), \tag{6}$$

and the claim follows since minimum of the lower bound (achieved for $n = 3$) is equal to $0.001565\ldots$, and this lower bound goes to infinity exponentially as $n \to \infty$. This lower bound can be improved to 0.089.

An action with the smallest Fried average entropy

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Inspection of the number fields data at http://www.lmfdb.org/ identifies the quartic totally real number field of discriminant 725 as the field that minimizes h^*_α. For it $h^*_\alpha = 0.330027... = h_{min}$.

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Hilbert

Siegel

Artin Representations

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Normalized defining polynomial

$x^4 - x^3 - 3x^2 + x + 1$

Invariants

Degree:	4
Signature:	$[4, 0]$
Discriminant:	$725 = 5^2 \cdot 29$
Ramified primes:	$5, 29$

Integral basis (with respect to field generator a)

$1,\ a^3 - a^2 - 2a + 1,\ a,\ a^2 - a - 1$

Class group and class number

Trivial Abelian Group, order 1

Unit group

Rank:	3
Torsion generator:	-1
Fundamental units:	$a^3 - a^2 - 2a,\ a,\ a - 1$
Regulator:	0.825068847934757

Properties

Degree:	4
Signature:	$[4, 0]$
Discriminant:	$5^2 \cdot 29$
Ramified primes:	$5, 29$
Class number:	1
Class group:	Trivial
Galois Group:	D_4

Related objects

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Completeness of this data

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Conjecture

The Cartan action α corresponding to the quartic totally real number field of discriminant 725 and the defining polynomial $x^4 - x^3 - 3x^2 + x + 1$ minimizes the Fried average entropy $h^*(\alpha)$ among all Cartan actions α. For this action $h^*(\alpha) = 0.330027... = h_{min}$, and it is given by

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 \\ -1 & 1 & -1 & 2 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ -2 & 2 & -3 & 3 \\ 1 & -1 & 1 & -2 \end{pmatrix}.$$

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Suppose K is a totally real number field of degree n for which $h^*_\alpha < h_{min}$. Then $\frac{0.000376 \exp(0.9371n) 2^{n-1}}{\binom{2n-2}{n-1}} < 0.33002 \Rightarrow n \leq 16$.

Lemma

The Fried average entropy h^*_α of a Cartan action of rank $n-1$ for $3 \leq n \leq 7$ satisfies $h^*(\alpha) \geq h_{min} = 0.330027...$.

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We use Friedman's lower bound for the regulator of a totally real number field $[K:\mathbb{Q}] = n$ [F], $R_K > 2g(1/D_K)$, where

$$g(x) := \frac{1}{2^n 4\pi i} \int_{2-i\infty}^{2+i\infty} (\pi^n x)^{-s/2}(2s-1)\Gamma(\tfrac{s}{2})^n ds.$$

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[F] Eduardo Friedman, *Regulators and total positivity*, Publ. Mat. (2007), Proceedings of the Primeras Jornadas de Teor´õa de Números, 119–130.

[PZ] M. Pohst, H. Zassenhaus, *Algorithmic algebraic number theory*, Cambridge University Press, 1989.

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[V] J. Voight, *Enumeration of totally real number fields of bounded root discriminant.* Algorithmic number theory, 268–281, Lecture Notes in Comput. Sci., 5011, Springer, Berlin, 2008.

Improvement of the lower bound for Fried average entropy

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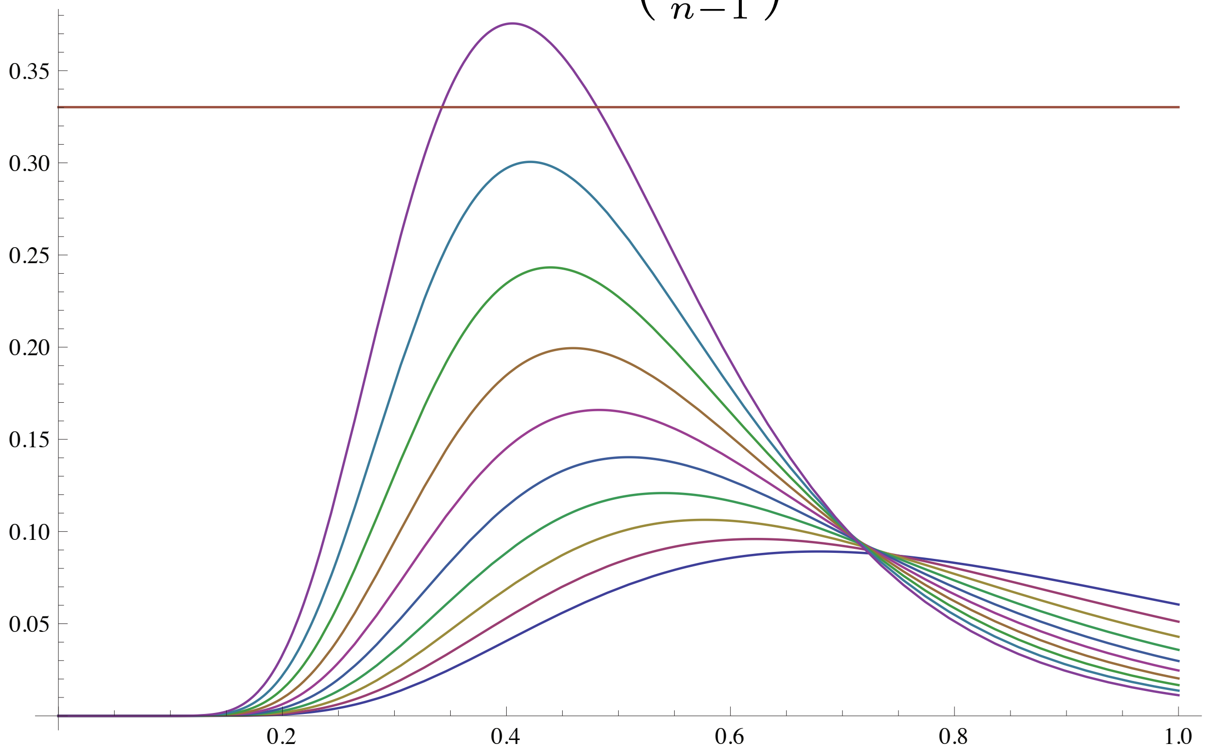
The following plots of $Z(n,s) = \frac{a(s)\exp(b(s))2^{n-1}}{\binom{2n-2}{n-1}}$ for $n = 8, \ldots, 16, 17$ show that



- a better bound than $n \le 16$ cannot be obtained from Zimmert's analytic formula;
- $h^*_\alpha \ge 0.089 = \min_n(\max_s Z(n,s))$.