Some number-theoretic tools used in homogenous dynamics

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I will start with a primer on algebraic number fields and define the notions of ring of integers, trace, norm, discriminant, group of units, and class number. I will present geometric representation of algebraic numbers, the logarithmic embedding, and the Dirichlet Unit Theorem, and will introduce a notion of regulator and methods for finding its lower bounds. I will explain the relation between commuting total automorphisms (n-by-n integral matrices with determinant ± 1) and algebraic units and will give several applications to dynamics.

For $n = 2$, there are natural bijections between conjugacy classes of hyperbolic elements in $SL(2, \mathbb{Z})$ of a given trace, closed geodesics of the same length on the modular surface, ideal classes in the corresponding real quadratic field, and congruence classes of primitive integral indefinite quadratic forms of the corresponding discriminant. If the class number of the corresponding quadratic field is greater than one, the non-isomorphic over $\mathbb{Z}$ matrices give rise to automorphisms of the 2-torus that are non-isomorphic algebraically but since these matrices are isomorphic over $\mathbb{Q}$, the entropies of the corresponding total automorphisms are equal, and, being Bernoulli, these automorphisms are measurably conjugate with respect to the Lebesgue measure. I will explain how to classify closed geodesics on the modular surface via Gauss reduction theory and continued fractions.

The situation for $n > 2$ is dramatically different due to a so-called measure rigidity. The counterparts of hyperbolic automorphisms of 2-torus are Cartan actions of $\mathbb{Z}^{n-1}$ on $\mathbb{T}^n$. They are generated by maximal rank abelian semisimple subgroups of $SL(n, \mathbb{Z})$. Measure rigidity for Cartan actions implies, in particular, that such actions are measurably conjugate only if they are algebraically conjugate over $\mathbb{Z}$. Accordingly, the algebraic data produce invariants of algebraic conjugacy. In particular, regulators of the corresponding number fields are closely related to a natural measure-theoretic notion of Fried average entropy.