PATTERNS IN PRIMES AND DYNAMICS ON NILMANIFOLDS

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Goal: Describe intertwining developments in ergodic theory, combinatorics and number theory related to solving linear equations in subsets of integers.

1. First case: Understanding one equation in three variables

1.1. Number Theory.

Theorem 1 (Vingoradov 1937). *Every sufficiently large odd number is the sum of* 3 *primes.*

$$N = x_1 + x_2 + x_3$$

Theorem 2 (Van der Corput 1939). *The primes contain infinitely many* 3*-term arithmetic progressions.*

$$x_1 + x_2 = 2x_3$$

Both results used the circle method for proof. Define the function

$$f(\alpha) = \sum_{\substack{p \le N \\ p \in \mathbb{P}}} e(p\alpha) = \sum \mathbb{1}_{\mathbb{P}}(x) e(\alpha x)$$

The number of solution to $x_1 + x_2 = 2x_3$ with $x_1 \le N$ is

$$\int_{\alpha \in \mathbb{T}} (f(\alpha))^2 \overline{f(2\alpha)} \mathrm{d}\alpha = \int e(\alpha(p_1 + p_2 - 2p_3)) \mathrm{d}\alpha = 1$$

if and only if $p_1 + p_2 - 2p_3 = 0$. Now, $f(0) = \sum_{p \le N} \mathbb{1}_{\mathbb{P}} \approx \frac{N}{\log N}$, so the contribution from an interval of size $\frac{1}{N}$ around zero is roughly

$$\left(\frac{N}{\log N}\right)^3 \cdot \frac{1}{N} = \frac{N^2}{(\log N)^3}.$$

We can get an asymptotic formula using results on primes in arithmetic progressions.

1.2. Combinatorics.

Theorem 3 (Roth 1953). *Let $E \subset [N] = 1, 2, \dots, N$, with $|E| = \delta N$ for $\delta > 0$, then for N large enough, E contains 3-term arithmetic progressions.*

Apply the same idea. Set

$$f(\alpha) = \sum_{x \le N} \mathbb{1}_E(x) e(\alpha x)$$

Then the number of 3-term arithmetic progression in E is

$$\int (f(\alpha))^2 \overline{f(2\alpha)} \mathrm{d}\alpha$$

Now, $f(0) = \delta N$, so if we take a small $(\frac{1}{N})$ interval around 0, then $\frac{(\delta N)^3}{N} = \delta^3 N^2$.

The argument proceeds as follows. Think of $E \subset P$, where P is an arithmetic progression of $|P| = N$ then:

- either E has lots, $> \frac{\delta^3 N^2}{2}$, of of 3-term progressions

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- or there is a nontrivial contribution from α not $\frac{1}{N}$-close to zero. There exists α such that

$$\left|\sum_{x\leq N}\left(\mathbb{1}_E(x)-\delta\right)e(\alpha x)\right| \gg_\delta N$$

In the second case, we use the equidistribution of $\{\alpha x\}_{x\leq N}$ to find a sub progression $P' \subset P$ with $|P'| = N^{1/3}$ such that

$$\frac{|E\cap P'|}{|P'|} > \delta + c(\delta)$$

after finitely many steps you end up with either many progressions, or a long progression of size N^α.

1.3. **Ergodic Theory.** An ergodic theoretic approach to Roth's Theorem, due to Furstenberg.

Observation 1 (Furstenberg Correspondence Principle)**.** Let $E \subseteq \mathbb{Z}$ of positive upper density, that is

$$\overline{\lim}\left|\frac{E\cap[N]}{N}\right| = \delta > 0,$$

then there exists a measure preserving system $(X, \mathcal{B}, \mu, T)$, μ T-invariant, and a set A of positive measure, such that if

$$\mu\left(A\cap T^{-n_1}A\cap\cdots\cap T^{-n_k}A\right) > 0 \Rightarrow E\cap E_{-n_1}\cap\cdots\cap E_{-n_k}\neq\emptyset$$

which means there exists an x such that $x, x+n_1, \ldots, x+n_k \in E$.

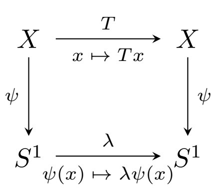
For Roth's Theorem, we need to find $n > 0$ such that $\mu\left(A\cap T^{-n}A\cap T^{-2n}A\right) > 0$. Assume X is ergodic, then we want to investigate the following average:

$$\frac{1}{N}\sum_{n\leq N}\mu\left(A\cap T^{-n}A\cap T^{-2n}A\right)$$

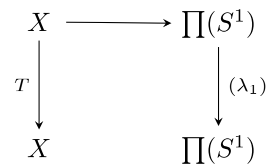
In this case we have that

- either
$$\frac{1}{N}\sum_{n\leq N}\mu\left(A\cap T^{-n}A\cap T^{-2n}A\right)\xrightarrow[n\to\infty]{}(\mu(A))^3$$
for all sets A with positive measure,
- or, G has a nontrivial eigenfunction.

If ψ is a nontrivial eigenfunction, $T\psi(x) = \lambda\psi(x)$, and $|\psi|$ is T-invariant. Without loss of generality, ψ takes values in S^1. We then get a morphism from X to a circle rotation system, $\psi : X \to S^2$, $x \mapsto \psi(x)$.



You can collect the contribution from all the ψ_i, normalized eigenfunctions, which gives a map, $\Pi : X \to \prod(S^1)$, from X to an abelian rotation system.



This is an example of a *Kronecker system*, the image, denoted $Z(X)$, is an abelian group and is called the *Kronecker factor* of X.

Theorem 4 (Furstenberg)**.** *Let X be an ergodic measure preserving system, and let $A \subset X$ be a set of positive measure. Consider the average*

$$\begin{aligned} \frac{1}{N}\sum_{n\leq N}\mu\left(A\cap T^{-n}A\cap T^{-2n}A\right) &= \frac{1}{N}\sum_{n\leq N}\int_X \mathbb{1}_A(x)\mathbb{1}_A(T^n x)\mathbb{1}_A(T^{2n}x) \\ &\sim \frac{1}{N}\sum_{n\leq N}\int \pi_*\mathbb{1}_A(z)\pi_*\mathbb{1}_A(z+n\alpha)\pi_*\mathbb{1}_A(z+2n\alpha)dz \quad (*) \end{aligned}$$

For $\alpha \in Z$, and where dz is the Haar measure on $Z(X)$.

Since we are in an abelian group,

$$(*) \to \int \pi_*\mathbb{1}_A(z)\pi_*\mathbb{1}_A(z+\beta)\pi_*\mathbb{1}_A(z+2\beta)\mathrm{d}z\mathrm{d}\beta > 0$$

In particular, if the projection $\pi_*\mathbb{1}_A$ is not trivial, we have

$$\left|\int \left(\pi_*\mathbb{1}_A - \mu(A)\right)\chi(z)\right| > 0$$

where χ is a nontrivial character.

2. Finding k-term arithmetic progressions

2.1. **Example.** The following system of linear equations corresponds to 4-term arithmetic progressions:

$$\begin{cases} x_1 + x_3 = 2x_2 \\ x_2 + x_4 = 2x_3 \end{cases}$$

This can be generalized to find systems of linear equations corresponding to k-term arithmetic progressions.

2.2. Combinatorics.

Theorem 5 (Szemerédi 1975)**.** *Let $E \subset \mathbb{Z}$ be a set of positive upper density, then E contains k-term arithmetic progressions for any k.*

Proof. Graph theoretic, does not generalize Roth's argument. □

2.3. Ergodic Theory.

Theorem 6 (Furstenberg 1977)**.** *Let $(X,\mathcal{B},\mu,T)$ be a measure preserving system, and $A \subset X$ with $\mu(A) > 0$, then there exists $n > 0$ such that*

$$\mu\left(A\cap T^{-n}A\cap\cdots\cap T^{-(k-1)n}A\right) > 0$$

Theorem 6 implies Theorem 5 by the Furstenberg correspondence principle.

For the proof we investigate $\frac{1}{N}\sum_{n\leq N}\mu\left(A\cap T^{-n}A\cap\cdots\cap T^{-(k-1)n}A\right)$. Furstenberg constructs a sequence of factors

$$X \to Z_{k-1} \to Z_{k-3} \to \cdots \to Z_1 \to \{*\}$$

where Z_1 is the Kronecker factor, satisfying the following:

- either $\pi_j : X \to Z_j$ is "relatively weakly mixing"
- or there is a morphism from X to an isometric extension of Z_j, $Z_j \times_\sigma M$, where $\sigma : Z_j \to \mathrm{Isom}(M)$.

In the second case we see that

$$\frac{1}{N}\sum_{n\leq N}\mu\left(A\cap T^{-n}A\cap\cdots\cap T^{-(k-1)n}A\right) \sim \frac{1}{N}\int \pi_{*_{k-2}}\mathbb{1}_A(z)\pi_{*_{k-2}}\mathbb{1}_A(Tz)\cdots\pi_{*_{k-2}}\mathbb{1}_A(T^{(k-1)n}z)\mathrm{d}\pi_{*_{k-2}}\mu$$

Theorem 7 (Furstenberg)**.** *The* lim inf *of the above average is positive.*

2.4. **Generalization of Roth's argument.** (Gowers)

Let $A \subset [N]$ have density δ, then

- either A has lots, $> \frac{N^2\delta^k}{2}$, of k-term progressions
- or there exists a partition of N into progressions P_i, $|P_i| \geq N^{\alpha_k(\delta)}$, such that

$$\sum_i \left| \sum_{x \in P_i} (\mathbb{1}_A(x) - \delta) e(p_i(x)\alpha) \right| \gg_\delta N$$

where $p_i(x)$ are polynomials of degree $k-2$.