ERGODIC PROPERTIES OF HOROCYCLIC FLOWS ON INFINITE VOLUME HYPERBOLIC SURFACES

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1. The Bowen-Margulis and Burger-Roblin measures

Recall our setting from yesterday. We have $\Lambda_\Gamma \subset S^1$ the limit set and $T^1S = \Gamma\backslash(S^1 \times S^1 \setminus \text{diag} \times \mathbb{R})$. We defined the sets $\Omega = \Gamma\backslash(\Lambda_\Gamma \times \Lambda_\Gamma \setminus \text{diag} \times \mathbb{R})$, which is not (h^s)-invariant, and $\mathcal{E} = \bigcup_s h^s\Omega = \Gamma\backslash(\Lambda_\Gamma \times S^1 \setminus \text{diag} \times \mathbb{R})$.

We also defined the Bowen-Margulis (Patterson-Sullivan) measure which was the measure of maximal entropy of (g^t) on Ω and $\tilde{m}_{BM} \sim \nu \times \nu \times \mathrm{d}t$, where ν is the Hausdorff measure of Λ_Γ. If the volume of S is finite, then $\tilde{\mathcal{L}} = \tilde{m}_{BM} \sim \lambda \times \lambda \times \mathrm{d}t$. This measure is not ν invariant under horocycle flow, so we constructed a new infinite (except in the finite volume case) measure $\tilde{m}_{BR} \sim \nu \times \lambda \times \mathrm{d}t$.

We can write $v = (v^-, v^+, t)$, then any vector on its orbit has coordinates $h^s v = (v^- t, w^+, t)$. The set of (h^s) orbits is equal to the set of $W^{su}(v)$ which is the unstable foliation of T^1S. Given B a chart of the foliation we have $m_{BM}\big|_B = \mu_T \cdot \mu)H$ and $m_{BR}\big|_B = \mu_T \cdot \mathrm{d}s$, where H is an unstable leaf, and T is the transverse direction. If the volume of S is finite, these measures are equal to the Liouville measure.

2. Proofs

2.1. Finite volume case.

In this setting we have that the Liouville measure decomposes into $\lambda \times \lambda \times \mathrm{d}t$, using Hopf's argument one can show that it is ergodic with respect to the geodesic flow. From this one can deduce the ergodicity of the horocycle flow and hence that the geodesic flow is mixing. One can then show that the Liouville measure is the unique h^s ergodic measure, except for periodic orbits, which then implies the equidistribution of non-periodic orbits.

2.2. Infinite volume case.

Assume m_{BM} is finite or Γ is finitely generated.

Theorem 1 (Sullivan). *If Γ is finitely generated then m_{BM} is finite.*

This theorem requires constant curvature, it is false in variable or negative curvature. It boils down to a computation on hyperbolic spaces.

Theorem 2 (Hopf argument). *If m_{BM} is finite, then it is ergodic.*

Use the fact that m_{BM} has a product structure. Consider any map and look at its ergodic averages. Apply the Birkhoff ergodic theorem and get that the ergodic averages converge positively and negatively to the conditional expectation. Project to the invariant sets, since the limits are the same positively and negatively, you can show that you get a map which is almost surely constant. This conclusion is false for non-ergodic measures, so you have to use the product structure and Fubini's theorem to complete the argument.

Theorem 3 (Rudolph, Babillot). *If m_{BM} is finite, then it is mixing.*

The proof is a refinement of Hopf's argument. First, assume it is not mixing and then find a contradiction. The product structure gives that the past and the future are independent.

Date: January 30, 2015.

Corollary 1. If m_{BM} is a finite probability measure, then for all $v \in \mathcal{E}, \varphi \in C_c(T^1S)$,

$$\int_{-1}^{1} (\varphi \circ g^t)(h^s v)\, \mathrm{d}\mu_H \longrightarrow \int \varphi \, \mathrm{d}m_{BM}$$

Thicken the piece of horocycle into a set A_ϵ, then you can rewrite the left had side as $\frac{1}{\epsilon}\int \varphi \cdot g^t \cdot \mathbb{1}_{A_\epsilon}\, \mathrm{d}m_{BM}$. Now use mixing to show that this converges to $\int \varphi \, \mathrm{d}m_{BM} \cdot \frac{m_{BM}(A_\epsilon)}{\epsilon}$

Theorem 4 (Roblin)**.** *If m_{BM} is finite, then m_{BR} is the unique (h^s)-invariant measure supported on the set $\{v^- \mid g^{-t}v$ returns infinitely often to a compact set$\}$*

Theorem 5 (Schapira)**.** *If Γ is finitely generated, then for all $v \in \mathcal{E}$, non (h^s) periodic, and for all $\varphi \in C_c(T^1S)$, we have*

$$\frac{\int_{(h^s v)_{|s|\le R}} \varphi \, d\mu_H}{\mu_H\left((h^s v)_{|s|\le R}\right)} \longrightarrow \frac{\int \varphi \, d_{m_{BM}}}{m_{BM}(T^1S)}$$

where the convergence is uniform on compact sets.

Theorem 6 (Maucourant-Schapira)**.** *For v and φ as above,*

$$\int_{-R}^{R} \varphi(h^s v)\, ds \sim R^\delta \tau(v,R) \int \varphi \, dm_{BR}$$

where $R^\delta \tau(v,R) = \mu((h^s v)_{|s|\le R})$.

Proof of Thm. 4. Let μ be an ergodic invariant conservative measure on

$$\{v \mid g^{-t}v \text{ returns infinitely often to a compact set}\}.$$

Let $v \in T^1S$ be generic for μ, we can find a sequence $\{t_k\}$ such that $g^{-t_k}v \longrightarrow v_\infty$, and V, a compact neighborhood of v_∞. Then for k large enough $(h^s g^{-t_k} v)_{|s|<1} \subset V$. Let $\varphi, \psi \in C_c(T^1S)$,

$$\frac{\int_{(h^s)_{|s|\le e^{t_k}}} \varphi(h^s v)\, \mathrm{d}\mu_H}{\int_{(h^s)_{|s|\le e^{t_k}}} \psi(h^s v)\, \mathrm{d}\mu_H} = \frac{\int_{(h^s g^{-t_k} v)_{|s|\le 1}} \varphi(g^{t_k} h^s g^{-t_k} v)\, \mathrm{d}\mu_H}{\int_{(h^s g^{-t_k} v)_{|s|\le 1}} \psi(g^{t_k} h^s g^{-t_k} v)\, \mathrm{d}\mu_H} \longrightarrow \frac{\int \varphi \, \mathrm{d}m_{BM}}{\int \psi \, \mathrm{d}m_{BM}}$$

from Theorem 3.

Let B be a chart of the foliation and $\varphi \in C_c(B)$. Consider the specific case when $\psi = \mathbb{1}_B$. Then to each φ we can associate the following ratio

$$\frac{\int_{(h^s v)_{|s|\le e^{t_k}}} \varphi(h^s v)\, \mathrm{d}\mu_H}{\int_{(h^s v)_{|s|\le e^{t_k}}} \mathbb{1}_B(h^s v)\, \mathrm{d}\mu_H} \simeq \int_T \left(\int_{\text{leaf}} \varphi \, \mathrm{d}\mu_H \right) \mathrm{d}\nu_{T,t_k},$$

where

$$\nu_{T,t_k} = \frac{\sum_{t \in T \cap (h^s v)_{|s|<e^{t_k}}} \delta_+}{\int_{(h^s v)_{|s|<e^{t_k}}} \mathbb{1}_B(h^s v)\, \mathrm{d}\mu_H} \longrightarrow \mu_T.$$

Since v is generic for μ, we get

$$\frac{\int_{(h^s v)_{|s|\le e^{t_k}}} \varphi(h^s v)\, \mathrm{d}s}{\int_{(h^s v)_{|s|\le e^{t_k}}} \mathbb{1}_B(h^s v)\, \mathrm{d}s} \longrightarrow \frac{\int \varphi \, \mathrm{d}\mu}{\int \mathbb{1}_B \, \mathrm{d}\mu}.$$

We can also write the denominator as

$$\int_T \left(\int_{\text{leaf}} \varphi(h^s v)\, \mathrm{d}s \right) \mathrm{d}\nu_{T,t_k} \times c(T,t_k),$$

since $\nu_{T,t_k} \longrightarrow \mu_T$ we get that the double integral converges to $\int \varphi \, \mathrm{d}m_{BR}$. From this we see that the constant, $c(T,t_k)$ cannot go to 0 and cannot go to ∞, and in fact it can be proven that the limit point must be constant. Thus we get our conclusion that $\mu = m_{BR}$, up to a normalization constant. □

Pf of Thm. 6. We follow a very similar strategy to the proof of theorem 4. Let $R = e^{t_k}$ and consider $\varphi \in C_c(T^1 S)$ and the averages

$$\frac{\int_{(h^s v)_{|s|\le R}} \varphi \, \mathrm{d}\mu_H}{\mu_H\left((h^s v)_{|s|\le R}\right)} \longrightarrow \int \varphi \, \mathrm{d}m_{BM},$$

by Thm. 5. Further assume that $\varphi \in C_c(B)$, where B is a chart of the foliation. Then we can rewrite the left hand side as

$$\int_T \left(\int_{\text{leaf}} \varphi \, \mathrm{d}\mu_H \right) \mathrm{d}\nu_{T,R},$$

where $\nu_{T,R}$ is as before. The right hand side can also be rewritten as

$$\int_T \left(\int_{\text{leaf}} \varphi \, \mathrm{d}\mu_H \right) \mathrm{d}\mu_T.$$

Taking these together implies that $\nu_{T,R} \longrightarrow \mu_T$.

Now consider the following ratios,

$$\frac{\int_{-R}^{R} \varphi(h^s v)\, \mathrm{d}s}{\int_{-R}^{R} \mathbb{1}_B(h^s v)\, \mathrm{d}s} = \int_T \left(\int_{\text{leaf}} \varphi(h^s v)\, \mathrm{d}s \right) \mathrm{d}\nu_{T,R} \times \frac{\mu_H\left((h^s v)_{|s|\le R}\right)}{\int_{-R}^{R} \mathbb{1}_B(h^s v)\, \mathrm{d}s}.$$

We can view the left hand side as a probability measure on B, which means the right hand side converges, up to subsequences, and thus

$$\frac{\mu_H\left((h^s v)_{|s|\le R}\right)}{\int_{-R}^{R} \mathbb{1}_B(h^s v)\, \mathrm{d}s} \longrightarrow c$$

and

$$\int_T \left(\int_{\text{leaf}} \varphi(h^s v)\, \mathrm{d}s \right) \mathrm{d}\nu_{T,R} \longrightarrow \int \varphi \, \mathrm{d}m_{BR}.$$

Thus we get the conclusion

$$\int_{-R}^{R} \varphi(h^s v)\, \mathrm{d}s \sim \mu_H\left((h^s v)_{|s|\le R}\right) c \int \varphi \, \mathrm{d}m_{BR}.$$

$\square$