HOMOGENEOUS FLOWS AND THE STATISTICS OF DIRECTIONS

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1. Introduction

1.1. **What is the statistics of directions?** Take a set $\mathcal{P}$ of points. Look at all the points which fall in a ball of radius T and consider their projection to the unit sphere, as in Fig. 1.

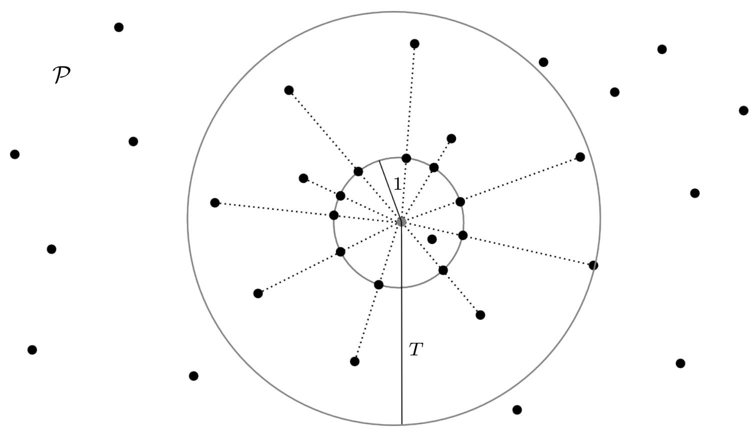

Figure 1. The projection of points within a ball of radius T to the unit sphere around the observer

There are several natural questions one can ask, such as:

(1) What happens as the radius T goes to infinity?
(2) What is the distribution of gaps between these projected points?
(3) What do the local statistics looks like?

1.2. **Examples.**

Example 1 (Quasicriystals)**.** You can use the set of vertices of an aperiodic tiling to get your point set. Baake, Göthe, Huck and Jakobi, examined this question for several sets of tilings. Numerically the distributions do not quite match up with the distribution from $\mathbb{Z}^2$.

Example 2 ($\mathcal{P} = \mathbb{Z}^2 \subset \mathbb{R}^n$)**.** The limiting distribution on angle spacings has been computed exactly by Boca, et al.

Example 3 ($\mathcal{P} \subset \mathbb{R}^2$ a random set)**.** This is a realization of a Poisson point process and the spacings have an exponential distribution.

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Example 4 (Other settings)**.**

$$\begin{aligned}\mathcal{P} &= \mathbb{Z}^d \text{ or } \mathcal{L}\\ &= \hat{\mathbb{Z}} \text{ primitive lattice points without multiplicity}\\ &= \mathbb{Z}^d+\alpha\\ &= \underline{e}\Gamma, \underline{e}\in\mathbb{R}^d\\ &= \text{ families of closed geodesics on translation surfaces - Athreya, Lelievre, Chaika}\\ &= \text{ hyperbolic lattice points}\end{aligned}$$

1.3. **Questions.** Let $\mathcal{P}_T = \mathcal{P}\cap B_T^d$, $0\notin\mathcal{P}$. There are several questions we want to explore

(1) How many elements are there in $\mathcal{P}_T$?
(2) As $T\to\infty$, does $\#\mathcal{P}_T \sim \Theta\text{vol}(B_T^d)$? (i.e. does it grow like a constant times the volume?) We call $\theta\text{vol}(B_T^d)$ the density of $\mathcal{P}$.
(3) Is $\left\{\frac{y}{\|y\|}\mid y\in\mathcal{P}_T\right\}$ uniformally distributed on S^{d-1}?
(4) Do the local statistics of the projection of $\mathcal{P}_T$ converge? For example, in $d=2$ we can look at the gap distribution. In general we can look at the distribution in small randomly placed discs, $\mathcal{D}_T^\sigma(v)\subset S_1^{d-1}$. Where v is the center of the disc, random according to a Borel probability measure $\lambda\ll\omega$, the natural volume measure on the unit sphere, σ is a fixed constant, and $\text{vol}(\mathcal{D}_T^\sigma(v)) = \frac{\sigma}{\theta T^d \text{vol} B^d}$. We would then like to know what the probability is of finding k points from our sequence in these discs.

2. Statistics of Directions

2.1. Setting.

Observation 1. Let $\mathcal{D}_T^\sigma(v)\subset S^{d-1}$ be one of the small discs. Then counting points that fall in this disc of radius approximately $T^{-\frac{d}{d-1}}$ is the same as counting points in the cone above the disk.

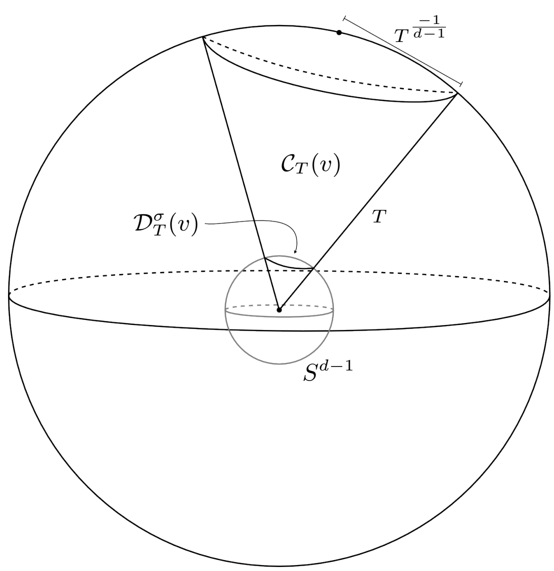

Figure 2. The cone $\mathcal{C}_T(v)$ above the disc $\mathcal{D}_T^\sigma(v)$.

2.2. **Homogeneous Flows.** We need to rescale the picture in the following way to make our cone well proportioned. Apply a rotation matrix $K: S_1^{d-1}\to \text{SO}(d)$ such that $vK(v)=e_1=(1,0,\dots,0)$. Set $D(T)=\text{diag}(T^{-1},T^{\frac{1}{d-1}},\dots T^{\frac{1}{d-1}})\in \text{SL}(d,\mathbb{R})$. Then $\mathcal{C}_T(v)K(v)D(T)=\mathcal{C}_T(e_1)D(T)\xrightarrow[T\to\infty]{}\mathcal{C}=$

$\{\underline{x} \in \mathbb{R}^d \mid 0 < x_1 < 1, \|(x_2, \dots, x_d)\| \leq x_1 \frac{\sigma}{\theta}\}$. Now our new question becomes counting $\#(\mathcal{P}K(v)D(T) \cap \mathcal{A})$, where $\mathcal{A}$ is a "nice" set, in this case $\mathcal{C}$. Choosing v random according to λ makes

$$\Xi_T = \mathcal{P}K(v)D(T)$$

a point process, i.e. a random point set. We would like to know whether this sequence of random point sequences has a limit.

Definition 1. Let Ξ_T be sequence of random point processes, and Ξ another point process. We say $\Xi_T \Rightarrow \Xi$ *in finite-dimensional distribution* if for all nice sets $\mathcal{A}_i \subset \mathbb{R}^d$ we have

$$\mathbb{P}(\#(\Xi_T \cap \mathcal{A}_1 = k_1, \dots \#(\Xi_T \cap \mathcal{A}_r) = k_r) \longrightarrow \mathbb{P}(\#(\Xi \cap \mathcal{A}_1) = k_1, \dots, \#(\Xi \cap \mathcal{A}_r) = k_r)$$

for all $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$.

3. Relative compactness

Define $E_T(k, \mathcal{A}) := \mathbb{P}(\#(\Xi_T \cap \mathcal{A}) = k)$. In this setting, relative compactness means that if we fix $\mathcal{A} \in \mathbb{R}^d$ be a bounded set, with $\mathrm{vol}(\partial\mathcal{A}) = 0$, and $\lambda \ll \omega$ where λ has bounded density, and assume $\sup_{T>0} \frac{\#\mathcal{P}_T}{\mathrm{vol} B_T^d} < \infty$. Then for all $\varepsilon > 0$, there exists k_ε such that for all $\mathcal{A} \subset B_R^d$

$$\sum_{k=k_\varepsilon}^{\infty} E_T(k, \mathcal{A}) < \epsilon.$$

Without loss of generality, we can assume $\lambda = \omega$.

To prove this statement, it suffices to prove

$$\sum_{k=0}^{\infty} k E_T(k, B_R^d) < \infty$$

which is just the expectation value of the number of points.

4. Trivial Examples

Fix $v_0 \in S_1^{d-1}$. Consider $\mathcal{P} = \{u^{1/d} v_0 \mid n \in \mathbb{N}\}$ then $\#\mathcal{P} = T^d + O(1)$.

Exercise: If

$$\lim_{T \to \infty} E_T(k, \mathcal{A}) = \begin{cases} 1 & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k > 0 \end{cases}$$

who that $\Xi_T \Rightarrow \Xi$ converges in finite distribution to a no-point process.

Note. $\liminf_{T\to\infty} \mathbb{E}(\#(\Xi_T \cap \mathcal{A})) > 0$, for the limiting no-point process this expectation is 0.