Introduction to Ratner's Theorems on Unipotent Flows

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Part 3: Completion of the Proof of an easy special case of Ratner's Measure-Classification Theorem

Measure-Classification Theorem [Furstenberg, Dani]

$G = \mathrm{SL}(2,\mathbb{R})$, Γ = lattice in G, $u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$,

μ = ergodic u^t-inv't probability measure on G/Γ

$\Longrightarrow$ μ is Lebesgue measure or on closed orbit.

Step 1 of the proof (yesterday)

Shearing: fastest motion is parallel to the orbits.

Prop. Fastest **transverse** motion is along norm'zer.

Cor. μ is inv't under $a^s = \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix}$ unless supported on closed u^t-orbit.

Step 2: μ is inv't under $\begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}$. *entropy calculation*

Step 2: Entropy calculation

What is entropy?

Fix **partition** $\mathcal{P} = \{A_1, \ldots, A_n\}$ of X. Let $p_i = \mu(A_i)$.

Suppose x is an <u>unknown</u> point in X.

Learning $x \in A_i$ gives us information:

bits of info is $\big|\log(\mu(A_i))\big| = |\log p_i|$.

Expected # bits info is $\sum_i p_i |\log p_i|$

= the **entropy** of $\mathcal{P}$ $= h_\mu(\mathcal{P}) \geq 0.$

For $T\colon X \to X$, **entropy** $h_\mu(T)$

= growth rate from $x, Tx, T^2x, \ldots, T^nx$.

$$\mathcal{P}_n = \{A_i\} \vee \{T^{-1}A_i\} \vee \cdots \vee \{T^{-n}A_i\}$$

$h_\mu(T) := \lim_{n\to\infty} h_\mu(\mathcal{P}_n)/n.$ (usually independent of $\mathcal{P}$)

$$h_\mu(T) := \lim_{n\to\infty} h_\mu(\mathcal{P}_n)/n \quad \geq 0$$

Eg. $T(x) = x + \alpha \pmod 1$ (with α irrational).
$[0,1) = [0,1/2) \cup [1/2,1)$
$\Rightarrow \#\mathcal{P}_n \leq 2 + \#\mathcal{P}_{n-1} \leq n$
$\Rightarrow h_{leb}(\mathcal{P}_n) \leq \log 2n$
$h_{leb}(T) \leq \lim_{n\to\infty} \frac{1}{n} \log 2n = 0.$

More generally:

Prop. $T \in \operatorname{Isom}(X) \implies h_\mu(T) = 0.$

no distances are stretched $\implies$ *entropy is 0*

amount of stretching = entropy for diffeos amount of stretching = entropy for diffeos

Pesin Entropy Formula [1977]

- T = *vol-pres diffeo of manifold* M (*cpct, smooth*)
- *tangent bundle* $\mathcal{T}M = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$ (T*-inv't*),

$$\forall \xi \in \mathcal{E}_i,\ \|T(\xi)\| = \tau_i \, \|\xi\|.$$

Then $h_{\mathrm{vol}}(T) = \sum_{\tau_i > 1} (\dim \mathcal{E}_i) \log \tau_i$.

Example (entropy of geodesic flow)

Recall $a^s q a^{-s} = \begin{bmatrix} \alpha & \beta e^{2s} \\ \gamma e^{-2s} & \delta \end{bmatrix}$. $\quad h_{\mathrm{vol}}(a^s) = 2|s|$

$$\mathcal{T}M = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}$$

Theorem (Ledrappier-Young [1985])

- T = *meas-pres diffeo of manifold* M
- *tangent bundle* $\mathcal{T}M = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$ (T-inv't),
  $\forall \xi \in \mathcal{E}_i, \|T(\xi)\| = \tau_i \|\xi\|$.
- H*-orbits are tangent to* $\bigoplus_{\tau_i > 1} \mathcal{E}_i$
- $\eta = \sum_{\tau_i > 1} (\dim \mathcal{E}_i) \log \tau_i$

Then $h_\mu(T) \le \eta$. *Equality* $\Leftrightarrow$ μ *is* H*-inv't.*

"measure of maximal entropy is nice"

Idea. If $\operatorname{supp} \mu$ misses directions that are stretched, they do not contribute as much as they should.
To exploit all directions (along the H-orbits), μ must be Lebesgue on every H-orbit.
So μ is H-invariant.

Theorem (Ledrappier-Young [1985])

If H-orbits tangent to the expanding directions of T, then $h_\mu(T) \le \eta$ = total stretching.

Equality $\Leftrightarrow$ μ is H-inv't.

Cor. Suppose μ is a^s-inv't on $\mathrm{SL}(2,\mathbb{R})/\Gamma$.
Then $h_\mu(a^s) \le 2|s|$, with equality iff μ is u^t-inv't.

Step 2. μ inv't under u^t and a^s $\implies$ $\mu =$ Lebesgue.

Proof.

μ is u^t-inv't $\implies$ $h_\mu(a^s) = 2|s|$ $\implies$ $h_\mu(a^{-s}) = 2|s|$

$\implies$ μ is invariant under $\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} = v^r$.

μ is invariant under $\langle u^t, a^s, v^r \rangle = \mathrm{SL}(2,\mathbb{R})$. □

Measure-classification ⇒ Equidistribution

Show $M_T(f) := \frac{1}{T}\int_0^T f(u^t x)\,dt \to \int_{Sx} f\,d\,\mathrm{vol} \quad \exists S$

Measure-Classification.

- Each ergodic measure is vol_{Sy}.
- Every inv't meas is $(\approx)$ $\sum_i \mathrm{vol}_{S_i y_i}$.

Recall that $M_T \in \mathrm{Meas}\,X$ and $\mathrm{Meas}(X)$ is compact.
Need to show only acc pt M_∞ of $\{M_T\}$ is some vol_{Sy}.

Key to Proof. Show $M_\infty(Sy) \neq 0 \Rightarrow \{u^t x\} \subseteq Sy$.

$\therefore \{u^t x\} \subseteq Sx$ and $\dim S$ minimal $\Rightarrow M_\infty = \mathrm{vol}_{Sx}$.

Claim. $d(u^t x, Sy)^2$ is polynomial function of t.

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Taylor series: $\log u = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k} (u - I)^k$

So $u^t = \exp(t \log u) = \sum_{k=1}^{n} \frac{1}{k!} t^k (\log u)^k$.

Each matrix entry of u^t is polynomial function.

Linearization (Dani-Margulis [1993])

Can show $S \doteq \overline{\overline{S}}$, so

$\exists$ homo $\rho \colon G \to \mathrm{SL}(D, \mathbb{R})$, and $\vec{v} \in \mathbb{R}^D$,

such that $S = \mathrm{Stab}_G(\vec{v})$. (Chevalley's Theorem)

Write $x = g\Gamma$, and assume $y = e\Gamma$.

$$d(u^t x, Sy)^2 \doteq d(u^t g, S)^2 \doteq d(u^t g v, v)^2.$$

u^t is polynomial func of t $\Rightarrow$ $u^t g v$ is polynomial

$\Rightarrow d(u^t g v, v)^2$ is polynomial.

Key to Proof. Show $M_\infty(Sy) \neq 0 \Rightarrow \{u^t x\} \subseteq Sy$.

We know $d(u^t x, Sy)^2$ is poly func of t of degree N.

$f \in C_c(X)_{\leq 1}$, supported in δ-neigh of Sy,
such that $M_T(f) > 0.01$.

$$
\begin{aligned}
0.01 < M_T(f) &= \frac{1}{T}\int_0^T f(u^t x)\,dt \\
&\leq \frac{1}{T}\ell(\{\, t \mid f(u^t x) \neq 0 \,\}) \\
&\leq \frac{1}{T}\ell(\{\, t \mid d(u^t x, Sy) < \delta \,\})
\end{aligned}
$$

$d(u^t x, Sy)^2$ is poly that is $< \delta$ on 1% of $[0, T]$
$\Rightarrow d(u^t x, Sy)^2 < \epsilon$ on $[0, T]$.

Let $T_k \to \infty$: $d(u^t x, Sy) = 0$ for all t.
So $\{u^t x\} \subseteq Sy$. □