Introduction to Ratner's Theorems on Unipotent Flows

Dave Witte Morris

University of Lethbridge, Alberta, Canada
http://people.uleth.ca/~dave.morris
Dave.Morris@uleth.ca

Part 2: Variations of Ratner's Theorem and Additional Ideas of the Proof

Ratner's Theorem [1991]

- G = Lie group = closed subgroup of $\mathrm{SL}(n, \mathbb{R})$
- Γ = lattice in G = discrete subgroup with $\mathrm{vol}(G/\Gamma) < \infty$
- $X = G/\Gamma$
- H = subgroup of G **gen'd by unips** $\subseteq \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$\Rightarrow \overline{Hx} = Sx$ for some subgroup S of G.

Conjecture (Shah [1998])

Suffices to assume ***Zariski closure*** $\overline{\overline{H}}$ *gen'd by unips*
$\approx$ *smallest almost conn subgrp of* G *that* $\supseteq H$.

Progress by Benoist-Quint [2013]

True if $\overline{\overline{H}} = \mathrm{SL}(n, \mathbb{R})$ (or semisimple) and H/H° is finitely gen'd

Ratner's Theorem

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$\Rightarrow \overline{Hx} = Sx$ for some subgroup S of G.

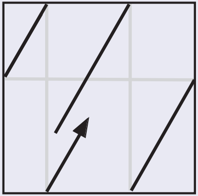
Derived from special case where $H = \{u^t\}$ is 1-dim'l

Ratner's Theorem

$\{u^t\}$ unipotent $\Rightarrow \overline{\{u^t\}x} = Sx$, $\exists$ subgrp S of G.

Also: $S \supseteq \{u^t\}$ and Sx has finite S-inv't volume.

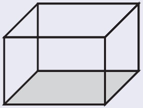
Example



Let X = torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.
Any $v \in \mathbb{R}^2$ defines a flow on $\mathbb{T}^2$:

$$x_t = x + tv \qquad \doteq u^t x$$

If the slope of v is irrational,
it is classical that every orbit is dense (& unif dist).
Also: Lebesgue is the only inv't probability measure.



$v = (a, b, 0)$ defines a flow on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$
and $\mathbb{T}^2 \times \{0\}$ is invariant.
Lebesgue on $\mathbb{T}^2 \times \{0\}$ is an invariant measure.

$\forall v$, any ergodic inv't probability meas for flow on $\mathbb{T}^3$ is Lebesgue meas on some subtorus $\mathbb{T}^k$ $\quad (0 \le k \le 3)$.

Ratner's Theorem on Orbit Closures

$\{u^t\}$ unipotent $\Rightarrow \overline{\{u^t\}x} = Sx$, $\exists$ subgrp S of G.

Ratner's Equidistribution Theorem

$\{u^t\}x$ is ~~dense~~ **equidistributed** in Sx:

$$\frac{1}{T}\int_0^T f(u^t x)\,dt \to \int_{Sx} f\,d\,\mathrm{vol} \quad \text{for } f \in C_c(G/\Gamma)$$

where vol = S-invariant volume form on Sx.

Ratner's Measure-Classification Theorem

Any ergodic u^t-invariant probability measure on G/Γ is S-invariant volume form on some Sx.

Generalized to p-adic groups. [Ratner, Margulis-Tomanov]

¿ characteristic p ? [Einsiedler, Ghosh, Mohammadi]

measure-classification $\Rightarrow$ equidist $\Rightarrow$ orbit-closure measure-classification ⇒ equidistribution

$$\frac{1}{T}\int_0^T f(u^t x)\,dt \to \int_{Sx} f\,d\,\mathrm{vol}$$

Easy case

μ = **unique** u^t-inv't probability meas on X (compact)
⇒ every u^t-orbit is equidistributed.

Proof. $M_T(f) := \frac{1}{T}\int_0^T f(u^t x)\,dt$
$M_T \colon C_c(X) \to \mathbb{C}$ positive linear functional

Riesz Rep Thm: $M_T \in \mathrm{Meas}(X) = \{\text{prob meas on } X\}$.

Banach-Alaoglu: $\mathrm{Meas}(X)$ weak* compact
⇒ subsequence $M_{T_n} \to M_\infty$ u^t-invariant.

Therefore $M_\infty = \mu$. So $M_T(f) \to \mu(f) = \int f\,d\mu$. □

Example: $G = \mathrm{SL}(2,\mathbb{R})$

Theorem (Furstenberg [1973])

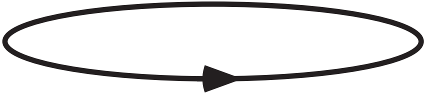
$\mathrm{SL}(2,\mathbb{R})/\Gamma$ *compact* $\Rightarrow$ *the only invariant probability measure for* $u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ *is the ordinary volume.*

Corollary

Every u^t*-orbit is equidistributed* ($\therefore$ *dense*).

Theorem (Dani [1978])

$\mathrm{SL}(2,\mathbb{R})/\Gamma$ *noncompact* $\Rightarrow$ *other* *ergodic* u^t*-invariant measure is the measure supported on a closed orbit.*

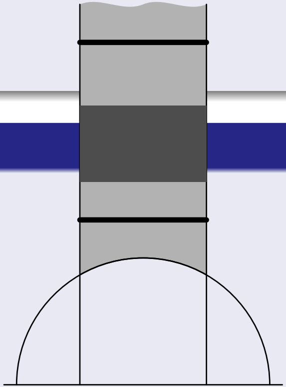


Theorem (Dani [1978])

$SL(2,\mathbb{R})/\Gamma$ *noncompact* ⇒ *other ergodic* u^t*-invariant measure is the measure supported on a closed orbit.*

General fact

Every invariant measure is a combination of ergodic measures.



Corollary

Every u^t*-inv't probability meas on* $SL(2,\mathbb{R})/\Gamma$ *is a combination of Lebesgue measure with measures on cylinders of closed orbits.*

Ratner's Measure-Classification Theorem

Ergodic u^t-inv't prob meas is S-inv't vol on Sx $(\exists S, x)$

Fix x_0. Sx_0 has finite S-inv't volume
$\Leftrightarrow$ gSx_0 has finite gSg^{-1}-inv't volume.
If $u^t \subseteq gSg^{-1}$, this provides a u^t-invariant measure.

- gSx_0 is analogue of a closed orbit.

$N(u^t, S) := \{ g \in G \mid u^t \subseteq gSg^{-1} \}$ ($\sim$submanifold).
$N(u^t, S)x_0 \subseteq G/\Gamma$ $\sim$ cylinder of closed orbits:
each gSx_0 has an u^t-invariant prob meas.

Lem. $\exists$ only countably many possible subgroups S.

Cor. Every u^t-inv't prob meas on G/Γ is a combo of measures on these countably many "tubes."

What does "ergodic" mean?

Defn. μ **ergodic** u^t-inv't meas:
every u^t-inv't meas'ble func is constant (a.e.)

Pointwise Ergodic Theorem

μ ergodic $\Leftrightarrow$ a.e. u^t-orbit is μ-equidistributed.

$$\frac{1}{T}\int_0^T f(u^t x)\,dt \to \int_X f\,d\mu \quad \text{a.e.}$$

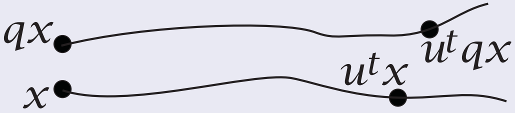
Exer. $\mu_1 \neq \mu_2$, both ergodic
$\Rightarrow \mu_1$ and μ_2 are mutually singular.

$$\mu_1(C_1) = 1,\ \mu_2(C_2) = 1,\ C_1 \cap C_2 = \varnothing$$

Hint. $\mu_1 = f\mu_2 + \mu_{sing}$ and f is u^t-inv't.

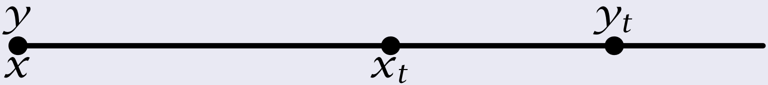
Proof of Measure-Classification

$$G = \mathrm{SL}(2,\mathbb{R}), \quad u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad a^s = \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix}.$$



$$u^t q u^{-t} = \begin{bmatrix} \alpha + \gamma t & \beta + (\delta - \alpha)t - \gamma t^2 \\ \gamma & \delta - \gamma t \end{bmatrix}$$

Shearing: Fastest motion is parallel to the orbits.



Shearing

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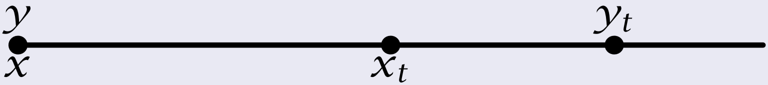
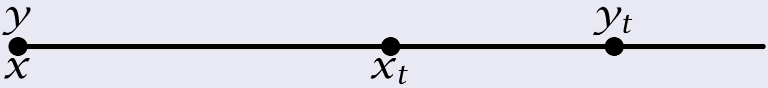


Contrast: $a^s q a^{-s} = \begin{bmatrix} \alpha & \beta e^{2s} \\ \gamma e^{-2s} & \delta \end{bmatrix}$

Fastest motion is transverse to the orbits.

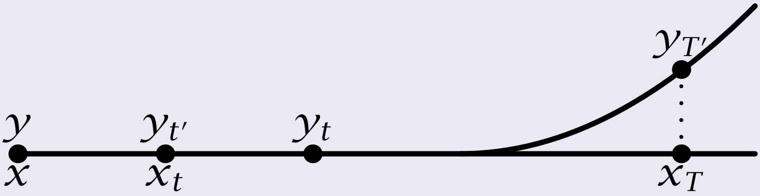
Shearing

Fastest motion is parallel to the orbits.



Key idea in proof of Measure-Classification

Ignore motion along the orbit, and look at the *transverse* motion perpendicular to the orbit.



Example

$$u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \; u^t q u^{-t} = \begin{bmatrix} \alpha + \gamma t & \beta + (\delta - \alpha)t - \gamma t^2 \\ \gamma & \delta - \gamma t \end{bmatrix}$$

Fastest motion is along $\{u^t\}$.

Ignoring this, largest terms are diagonal (in $\{a^s\}$)

Observation

$a^s \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} a^{-s} = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$: $\quad a^s$ **normalizes** $\{u^t\}$.

Proposition

For action of a unipotent subgroup, the fastest transverse divergence is along the ***normalizer****.*

Prop. Fastest transverse div is along **normalizer.**

Corollary (Step 1 of Ratner's Proof)

μ is u^t-inv't and ergodic (*and ...*) $\Rightarrow$ *μ is a^s-inv't.*

Proof.

a^s normalizes u^t $\Rightarrow u^t(a^s\mu) = a^s(u^{t'}\mu) = a^s\mu$.

μ and $a^s\mu$ are two *different* ergodic measures

$\Rightarrow$ live on disjoint u^t-invariant sets C and a^sC.

Assume $d(C, a^sC) > \epsilon$.

For $x \approx y$ in C: $C \ni u^t x \approx a^s u^{t'} y \in a^s C$ $(\exists t, t')$.

$\Rightarrow d(C, a^sC) < \epsilon$. $\rightarrow\leftarrow$ $\square$

Step 2: entropy calculation $\Rightarrow \mu$ inv't under $\begin{bmatrix} 1 & \\ * & 1 \end{bmatrix}$.