Quadratic Weyl sums, Automorphic Functions, and Invariance Principles

Francesco Cellarosi (UIUC)

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Plan of the talk

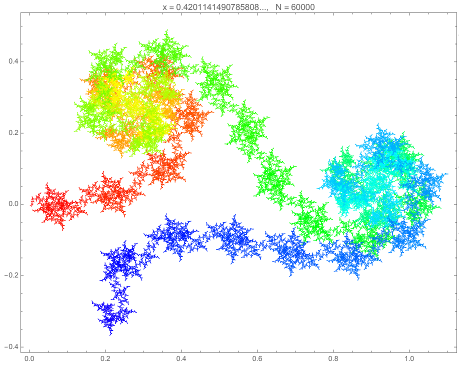
- Motivation: Hardy and Littlewood's 1914 paper on exponential sums and their approximate functional equation for theta sums.
- A new approximate functional equation.
- Invariance principle for quadratic Weyl sums
- An outline of how homogeneous dynamics is used

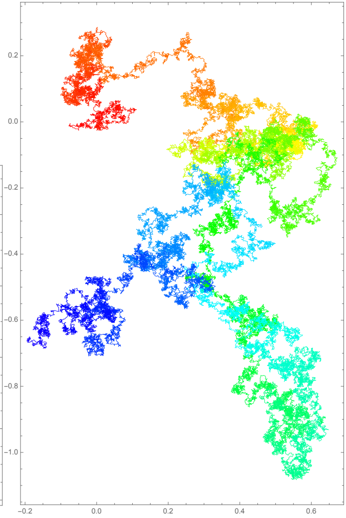
Randomness in number theory: a disclaimer

Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin.

(John von Neumann, 1951)

Curlicues v. Brownian motion





Jacobi theta function

Consider the classical Jacobi's elliptic theta function

$$\vartheta(z,w)=\sum_{n\in\mathbb{Z}} e\left(\tfrac{1}{2}n^2 z+nw\right),$$

where $e(x):=e^{2\pi i x}$, $z\in\mathcal{H}=\{z\in\mathbb{C}:\mathrm{Im}(z)>0\}$, $w\in\mathbb{C}$,

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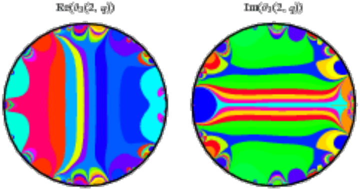

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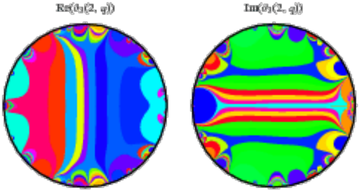

which satisfies the exact functional equation

$$\vartheta(z,w)=\sqrt{\frac{i}{z}}\,e\left(-\frac{w^2}{z}\right)\vartheta\left(-\frac{1}{z},\frac{w}{z}\right).$$

Hardy and Littlewood's 1914 paper 1/2

G.H. Hardy and J.E. Littlewood (1914) studied the theta sum (quadratic Weyl sum)

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$$S_N(x,\alpha)=\sqrt{\frac{i}{x}}\,e\left(-\frac{\alpha^2}{x}\right)S_{\lfloor xN\rfloor}\left(-\frac{1}{x},\frac{\alpha}{x}\right)+O\left(\frac{1}{\sqrt{x}}\right).$$

valid for $0<x<2$, $0\leq\alpha\leq 1$.

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valid for $0 < x < 2$, $0 \le \alpha \le 1$.
It is enough to consider $0 < x < 1$. We have a renormalization formula, which can be iterated...

Hardy and Littlewood's 1914 paper 2/2

Iterating the approximate functional equation for $S_N(x, \alpha)$ we can get estimates in terms of the continued fraction expansion of $x = [a_1, a_2, a_3, \ldots]$.

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Theorem A (Hardy-Littlewood)

- If x is of bounded type, then $S_N(x,\alpha)=O(\sqrt{N})$.
- If $a_n=O(n^\rho)$, then $S_N(x,\alpha)=O\left(N^{\frac{1}{2}}(\log N)^{\frac{\rho}{2}}\right)$.
- If $a_n=O(e^{\sigma n})$ and $\sigma<\frac{\log 2}{2}$, then $S_N(x,\alpha)=O\left(N^{\frac{1}{2}+\frac{\sigma}{\log 2}+\varepsilon}\right)$ for every $\varepsilon>0$.
- For almost every x, $S_N(x,\alpha)=O\left(N^{\frac{1}{2}}(\log N)^{\frac{1}{2}+\varepsilon}\right)$ for every $\varepsilon>0$.

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- If $a_n=O(e^{\sigma n})$ and $\sigma<\frac{\log 2}{2}$, then $S_N(x,\alpha)=O\left(N^{\frac12+\frac{\sigma}{\log 2}+\varepsilon}\right)$ for every $\varepsilon>0$.
- For almost every x, $S_N(x,\alpha)=O\left(N^{\frac12}(\log N)^{\frac12+\varepsilon}\right)$ for every $\varepsilon>0$.

Theorem B (Fiedler-Jurkat-Körner / Flaminio-Forni)
For almost every x there is a full measure set of α so that $S_N(x,\alpha)=O\left(N^{\frac12}(\log N)^{\frac14+\varepsilon}\right)$ for every $\varepsilon>0$.

A new approximate functional equation 1/2

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$$H_- = \{g \in G : \ \Phi^{-s} g \Phi^{s} \to e \ \text{ as } s \to \infty\} = \left\{\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \end{pmatrix}, 0\right)\right\}$$
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We have $H_+ = \{n_+(x,\alpha)\} \cong \mathbb{R}^2$ and $H_- = \{n_-(u,\beta)\} \cong \mathbb{R}^2$.

A new approximate functional equation 2/2

Theorem 1 (C.-Marklof)
There exist a cofinite $\Gamma < G$ and a square-integrable function $\Theta : \Gamma\backslash G \to \mathbb{C}$ and, **for every** $x \in \mathbb{R}$, a measurable function $E^x : H_- \to [0, \infty)$ and a set $P^x \subset H_-$ of full measure, such that **for all** $x, \alpha \in \mathbb{R}$ and $n_-(u, \beta) \in P^x$,

$$\left| S_N(x, \alpha) - e^{s/4}\Theta(\Gamma n_+(x, \alpha) n_-(u, \beta)\Phi^s) \right| \le E^x(u, \beta),$$

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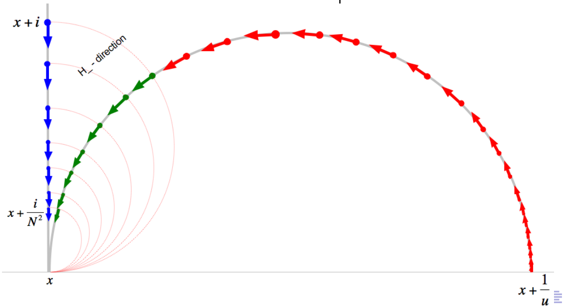

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- The set P^x is explicit (in terms of a Diophantine condition).

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- What do we mean by "random behavior"? Look at the "deterministic" random walk with increments $e(P(n)x + n\alpha)$...

A “deterministic” random walk

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- $S_N(x)$ is a sum of *strongly dependent* random variables. (Methods from Probability do not apply).

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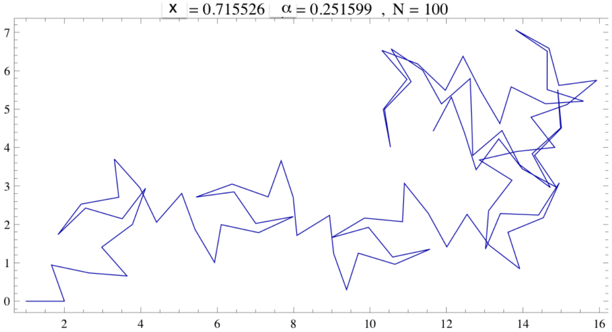

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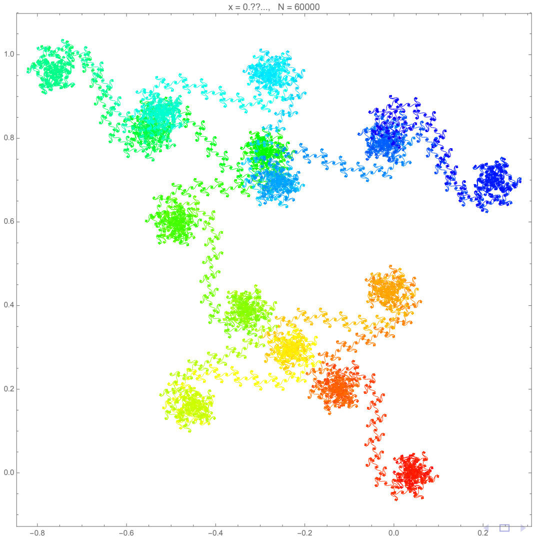
for $t \in [0,1]$. Notice that this is a curve of length $\sqrt{N}$ in $\mathbb{C}$.

A "deterministic" random walk - *curlicues*



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Theorem 2 (C.-Marklof).

- There exists a random process $t \mapsto X(t)$ such that $X_N(t) \Longrightarrow X(t)$ as $N \to \infty$.
- The process $t \mapsto X(t)$ does not depend on (c_1, α) or h.

Properties of the process $t \mapsto X(t)$

Theorem 2' (C.-Marklof).
The process $t \mapsto X(t)$ satisfies the following properties:

- **Tail asymptotics** (+ power saving).

$$\mathbb{P}\{|X(1)| > R\} = \frac{6}{\pi^2} R^{-6} \left(1 + O(R^{-\frac{12}{31}})\right).$$

- **Increments**. For every $t_0 < t_1 < \ldots < t_k$ the increments

$$X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_k) - X(t_{k-1})$$

  are not independent.
- **Scaling** For $a > 0$ let $Y(t) = \frac{1}{a} X(a^2 t)$. Then $Y \sim X$.

Properties of the process $t \mapsto X(t)$

- **Time inversion**. Let

$$Y(t) := \begin{cases} 0 & \text{if } t = 0; \\ tX(1/t) & \text{if } t > 0. \end{cases}$$

  Then $Y \sim X$.
- **Law of large numbers**. Almost surely, $\lim_{t\to\infty} \frac{X(t)}{t} = 0$.
- **Stationarity**. For $t_0 \geq 0$ let $Y(t) = X(t_0 + t) - X(t_0)$. Then $Y \sim X$.
- **Rotational invariance**. For $\theta \in \mathbb{R}$ let $Y(t) = e^{2\pi i\theta} X(t)$. Then $Y \sim X$.

Properties of the process $t \mapsto X(t)$

- **Modulus of continuity**. For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\limsup_{h \downarrow 0} \sup_{0 \le t \le 1-h} \frac{|X(t+h) - X(t)|}{\sqrt{h}(\log(1/h))^{1/4+\varepsilon}} \le C_\varepsilon$$

  almost surely.
- **Hölder continuity**. Fix $\theta < 1/2$. Then, almost surely, the curve $t \mapsto X(t)$ is everywhere locally θ-Hölder continuous.
- **Nondifferentiability**. Fix $t_0 \ge 0$. Then, almost surely, the curve $t \mapsto X(t)$ is not differentiable at t_0.

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- The limiting process $t \mapsto X(t)$ is the image of the geodesic flow on $\Gamma\backslash G$ (started at a Haar-random point) under a complex-valued function Θ (from **Thm 1**).
- The properties of $X(t)$ in **Thm 2'** come from properties of the geodesic flow on $\Gamma\backslash G$ (e.g. excursions into the cusp) and Θ.

An application

A particular case of **Thm 2'** is:

Theorem 2" (C. - Marklof) Fix $c_1, c_0, \alpha \in \mathbb{R}$, $(c_1, \alpha) \notin \mathbb{Q}^2$. There exists a probability measure $\mathbb{P}$ on $\mathbb{C}$ such that for every bounded continuous function $F : \mathbb{C} \to \mathbb{R}$ we have

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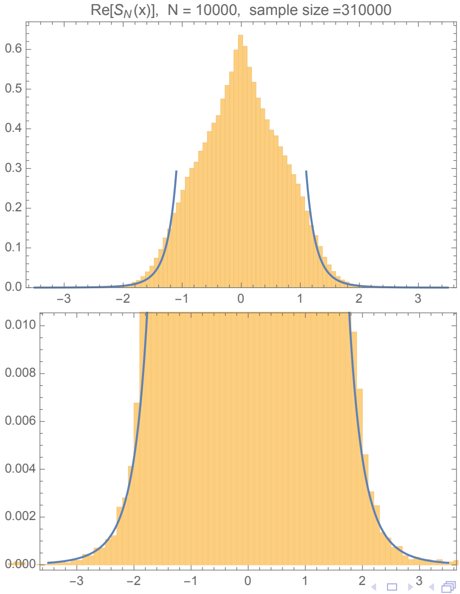
$$\int_{\mathbb{R}} F\left(N^{-\frac{1}{2}} S_N(x)\right) d\lambda(x) \overset{N\to\infty}{\longrightarrow} \int_{\mathbb{C}} F d\mathbb{P}.$$

Corollary. Fix $c_1, c_0, \alpha \in \mathbb{R}$, $(c_1, \alpha) \notin \mathbb{Q}^2$. Then

$$\lim_{N\to\infty} \lambda\left\{x \in \mathbb{R} : \left|\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \cos(2\pi(P(n)x + n\alpha))\right| > R\right\} =$$
$$= \frac{15}{16\pi^2} R^{-6} \left(1 + O(R^{-12/31})\right).$$


Re[S_N(x)], N = 10000, sample size =310000
0.6
0.5
0.4
0.3
0.2
0.1
0.0
−3
−2
−1
0
1
2
3
0.010
0.008
0.006
0.004
0.002
0.000
−3
−2
−1
0
1
2
3

Appendix: More on the tail asymptotics

$S_N(x) = \sum_{n=1}^{N} e(P(n)x + n\alpha), \quad P(n) = \frac{1}{2}n^2 + c_1 n + c_0.$

For $(c_1, \alpha) \notin \mathbb{Q}^2$ we have

$$\lim_{N\to\infty} \lambda\left\{x \in \mathbb{R} : \left|N^{-\frac{1}{2}} S_N(x)\right| > R\right\} = \frac{6}{\pi^2} R^{-6}\left(1 + O(R^{-12/31})\right)$$

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- Where does $\frac{6}{\pi^2}$ come from? In our proof

$$\frac{6}{\pi^2} = \frac{1}{\frac{\pi^2}{3}} \cdot \frac{2}{3} \cdot 2 \cdot \mathcal{I}$$

$$\mathcal{I} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \int_0^1 e(z_1 x^2 + z_2 x) dx \right|^6 dz_1 dz_2 = \frac{3}{2}$$

Counting integer points on Vinogradov's quadric 1/3

Let $\mathcal{N}(R)$ be the number of integer solutions to the equations

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3$$
$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2$$

with $1 \le x_j, y_j \le R$ for $j = 1, 2, 3$.

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- Hua (1959) also showed that the number $\tilde{\mathcal{N}}(a)$ of solutions with $x_1^2 + x_2^2 + x_3^2 \leq a$ and $y_1^2 + y_2^2 + y_3^2 \leq a$ is

$$\tilde{\mathcal{N}}(a) = \frac{35\sqrt{3}}{2} a^{3/2} \log a + O(a^{3/2}\sqrt{\log a}).$$

Counting integer points on Vinogradov's quadric 2/3

- Bykovskii (1984) showed that

$$\mathcal{N}(R) = \frac{12}{\pi^2}\,\mathcal{I}\,R^3 \log R + O(R^3)$$

$$\text{where} \quad \mathcal{I} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\int_0^1 e(z_1x^2 + z_2x)dx\right|^6 dz_1dz_2.$$

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- It was shown by V. Blomer and J. Brüdern (2010) that

$$\mathcal{N}(R) = \frac{18}{\pi^2} R^3 \log R + \frac{3}{\pi^2}\left(\gamma - 6\frac{\zeta'(2)}{\zeta(2)} - 5\right) R^3 + O(R^{5/2}\log R).$$

Counting integer points on Vinogradov's quadric 3/3

Rogovskaya's work implies that

$$\mathcal{I} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\int_0^1 e(z_1x^2 + z_2x)dx\right|^6 dz_1dz_2 = \frac{3}{2}$$

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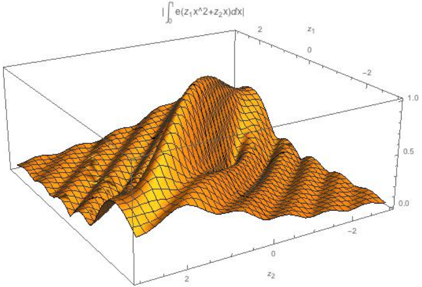

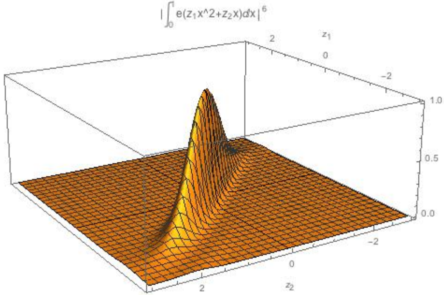

QUADRATIC WEYL SUMS, AUTHOMORPHIC FUNCTIONS, AND INVARIANCE PRINCIPLE

FRANCESCO CELLAROSI

1. Homogeneous Dynamics

1.1. **Setup.** We start with the Jacobi group $G = \widetilde{\mathrm{SL}}(2,\mathbb{R}) \ltimes \mathbb{H}(\mathbb{R})$. We can represent this as $\mathcal{H}\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}$, a six dimensional space with coordinates given by $(x+iy,\phi;\xi,\zeta)=g$. In these coordinates the Haar measure is $\mathrm{d}g = \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\phi\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\zeta}{y^2}$.

1.2. **Description of the limiting process $t \mapsto X(t)$.** In Theorem 2, we state that there exists a random process, but we can do better. We can actually give a description of this limiting process.

$$X(t) = \sqrt{t}\Theta(\Gamma g \Phi^{2\log t})$$

for g Haar-random on $\Gamma\backslash G$. We can think of $(\Gamma\backslash G, \mathrm{d}g)$ as a probability space. We need to understand Θ.

We have a Schrödinger - Weyl representation of G. To each element G we associate a unitary operator $U(L^2(\mathbb{R}))$, where $g\mapsto R(G)$, and $R(g)$ gives an operator $L^2\to L^2$, where $f\mapsto R(g)f$. We define $\Theta : G\to\mathbb{C}$ in terms of this representation as follows:

$$\Theta_f(g) = \sum_{n\in\mathbb{Z}}[R(g)f](n) = y^{1/4}e\left(\zeta - \frac{1}{2}\xi_1\xi_2\right)\sum_{n\in\mathbb{Z}} f_\phi\left((n-\xi_2)y^{1/2}\right)e\left(\frac{1}{2}(n-\xi_2)^2x + n\xi_1\right)$$

where $f_\phi(t) = \sum_{k=0}^\infty \hat{f}(k)e^{\frac{-i2k-1}{2}\phi}\psi_k(t)$, $\{\psi_k\}$ is a hermit orthonormal basis of $L^2(\mathbb{R})$, and $\hat{f}(k) = \langle f,\psi_k\rangle$. Note that when $\phi=0$, f_ϕ is the identity, if $\phi=\pi/2$, then f_ϕ is the Fourier transform of f.

Fact 1. *If f is, for example, Schwartz, then Θ_f is Γ-invariant for an explicit $\Gamma<G$. If f is not Schwartz, then we do not understand how to interpret Θ_f point wise.*

Observation 1 (Why we care about Θ_f)**.** When $y=\frac{1}{N^2}$ and $\chi = \mathbb{1}_{(0,1]}$, we have

$$S_N(x) = y^{-1/4}\Theta_\chi\left(x+iy, 0; \begin{pmatrix}\alpha + c_1x\\ 0\end{pmatrix}, c_0x\right).$$

Moreover,

$$X_N(t) = e^{s/4}\Theta_\chi\left(x+iye^{-s}, 0; \begin{pmatrix}\alpha + c_1x\\ 0\end{pmatrix}, c_0x\right),$$

$s = 2\log t$. We can use the group law to rewrite this as

$$e^{s/4}\Theta\left(\left(1;\begin{pmatrix}\alpha + c_1x\\ 0\end{pmatrix}, c_0x\right)\Psi^x\Phi^s\Phi^{2\log N}\right)$$

where $\Psi^x = \left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, 0\right)$, horocycle flow, and Φ is the geodesic flow.

Now, fix t, then we define $\widetilde{\Theta}(g) = e^{s/4}\Theta(g\Phi^s)$. The existence of finite dimensional limiting distribution means for $B:\mathbb{C}\to\mathbb{R}$

$$\lim_{\tau\to\infty}\int_{\mathbb{R}} B\left(\widetilde{\Theta}\left(\left(1;\begin{pmatrix}\alpha + c_1x\\ 0\end{pmatrix}, c_0x\right)\Psi^x\Phi^\tau\right)\right)\mathrm{d}\lambda(x) = \int_{\Gamma\backslash G} B\left(\widetilde{\Theta}(g)\right)\mathrm{d}g,$$

$\tau = 2\log N$, using the equidistribution of long closed horocycles in $\Gamma\backslash G$.

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