RIGIDITY OF HIGHER RANK DIAGONALIZABLE ACTIONS

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1. First examples

1.1. $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We can look at multiplication by $p \in \mathbb{Z}$, $p\times \curvearrowright \mathbb{T}$. If $p = 3$, we get that the middle third Cantor set is a $\times 3$ invariant set, modulo integers. Thus, there are many different types of orbits, closed invariant sets, and invariant measures.

1.2. $\mathbb{T}^2$. Consider the action $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$. Adler-Weiss gave a complete description of this dynamical system. We have a covering of $\mathbb{T}^2$ by a shift space in $\{0,1\}^{\mathbb{Z}}$. Again we get that there are many different types of orbits, closed invariant sets, and invariant measures

1.3. $X_2 = \mathbf{SL}(2,\mathbb{R})/\mathbf{SL}(2,\mathbb{Z})$. Where the action of $\mathrm{SL}(2,\mathbb{Z})$ is by Möbius transformations, that is $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$.

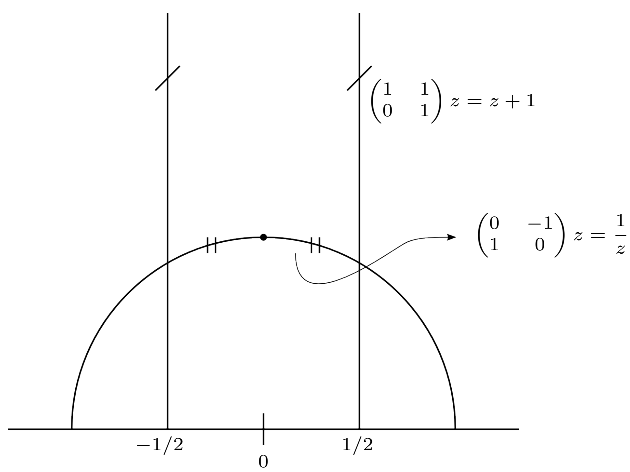

Figure 1. The space X_2 with identifications

Consider the action of the geodesic flow $A = \begin{pmatrix} e^2 & \\ & e^{-t} \end{pmatrix}$ by left multiplication on X_2. There is a relation between the continued fraction expansion and the geodesic flow. Consider a geodesic with right endpoint $\alpha \in \mathbb{R}$, shifting the geodesic gets us another geodesic with right endpoint at $\alpha - 1$, eventually we will reach $\{\alpha\}$, the fractional part of α, see Figure 2.

2. Higher rank

Theorem 1 (Furstenberg, '67). *Consider the actions $2\times, 3\times \curvearrowright \mathbb{T}$, and $\alpha \in \mathbb{T}$. We have*

$$S = \overline{\{2^n 3^m \alpha\}} = \begin{cases} \textit{finite} & \textit{if } \alpha \in \mathbb{Q} \\ \mathbb{T} & \textit{otherwise} \end{cases}$$

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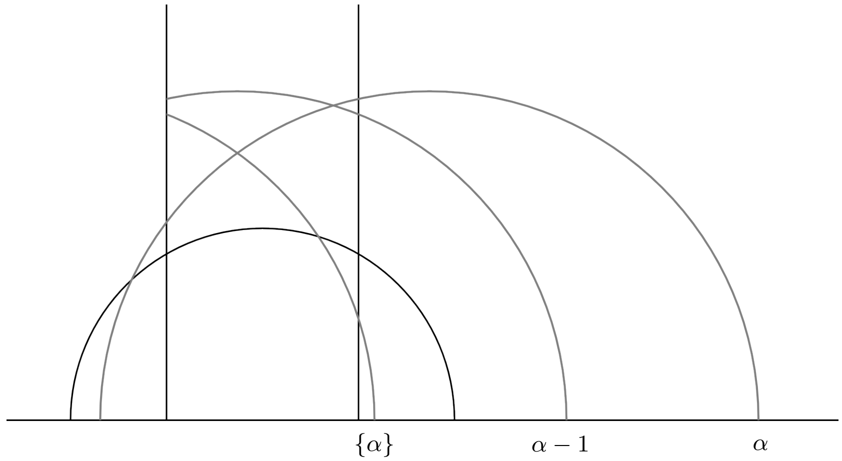

FIGURE 2. Relation between geodesic flow and continued fractions

Theorem 2 (Berend)**.** *Consider the actions $A, B : \mathbb{T}^k \to \mathbb{T}^k$. Assume these actions are totally irreducible, hyperbolic, and faithfully $\mathbb{Z}^2$. Take an A, B-invariant set S, then either $|S| < \infty$ or $S = \mathbb{T}$.*

3. Homogeneous Setting

We would like to get a higher rank picture so that we have 2 multiplicatively independent actions. Our space is $X_3 = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ where $\mathrm{SL}(3,\mathbb{Z}) = \{g\mathbb{Z}^3 \mid g \in \mathrm{SL}(3,\mathbb{R})$ and we consider the action of

$$A = \left\{ \begin{pmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{t^3} \end{pmatrix} \mid t_1 + t_2 + t_3 = 0 \right\}$$

Conjecture 1 (Margulis)**.** *If $Ax \subseteq X_3$ is bounded for some $x \in X_3$, then Ax is periodic.*

This is somehow the correct analog for the theorems of Furstenberg and Berend in this situation. It also gives the following:

Conjecture 2 (Littlewood's Conjecture)**.**

$$\liminf_{n\to\infty} n|||n\alpha_1||| \cdot ||||n\alpha_2||| = 0$$

4. Measure Rigidity

Theorem 3 (Rudolph '90)**.** *Suppose μ is $\times 2, \times 3$-invariant and ergodic, and $h_\mu(\times 2^m 3^n) > 0$ then $\mu = m =$ Lebesgue.*

Generalizations:

- Katok-Saptzier: $\mathbb{T}^k$ and G/Γ
- Kalinin-Katok: $A, B : \mathbb{T}^3$.

Theorem 4 (Einsiedler-Lindenstrauss, '04)**.** *Consider the actions of $A, B : \mathbb{T}^k \to \mathbb{T}^k$, totally irreducible, hyperbolic, and faithful. Then if $h_\mu(A^n B^m) > 0$ and μ invariant and ergodic, then $\mu = m_{\mathbb{T}^k}$.*

Theorem 5 (Lindenstrauss, '06)**.** *Consider the space $X_{2,p} = SL(2, \mathbb{R} \times \mathbb{Q}_p)/SL(2, \mathbb{Z}[\frac{1}{p}])$ and the action of*

$$A = \left\{ \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}, \begin{pmatrix} p^n & \\ & p^{-n} \end{pmatrix} \right\}$$

If μ is an A-invariant and ergodic measure, with $h_\mu(a) > 0$ then $\mu = m_{X_{2,p}}$.

Theorem 6 (EKL)**.** *Consider the action of A on X_3 and μ A-invariant and ergodic with $h_\mu(a) > 0$, then $\mu = m_{X_3}$. If we have A acting on X_n, then μ_n is homogeneous and if n is prime, $\mu = m_{X_n}$.*

Our lattice needs to be $\mathrm{SL}(3,\mathbb{Z})$ for the following reason:

Theorem 7 (Ree's example)**.** *There exists $\Gamma \subset SL(3,\mathbb{R})$, such that in $X = SL(3,\mathbb{R})/\Gamma$ we have a centralizer direction that closes up.*

Theorem 8 (EL)**.** *Let $X = \mathbb{G}(\mathbb{R} \times \mathbb{Q}_p)/\mathbb{G}(\mathbb{Z}[\frac{1}{p}])$ and consider the action of A, the maximal $\mathbb{R}/\mathbb{Q}_p$-split subgroup in some of the (almost) direct factors. Then there are 4 possible conclusions depending on your measure and lattice, either:*

- (I) *Invariance*
- (Z) *Zero entropy*
- (R) *Rank one*
- (T) *Periodic orbit*