SEMIGROUPS IN SEMISIMPLE GROUPS

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1. Density of eigenvalues

1.1. **Zariski dense semigroups.** Assume $V = \mathbb{R}^d$, $G = \mathrm{SL}(V) = \{g \in \mathrm{End}(V) \mid \det g < 1\}$. We define $\mathfrak{a} = \{x = (x_1, \dots, x_d) \mid x_1 + \dots + x_d = 0\}$ and the Weyl chamber $\mathfrak{a}_+ = \{x \in \mathfrak{a} \mid x_1 \geq \dots \geq x_d\}$. We then have a Jordan projection $\lambda : G \to \mathfrak{a}_+$ given by $\lambda(a) = (\log \lambda_1(g), \dots, \log \lambda_d(g))$ where $\lambda_i(g)$ are the moduli of the eigenvalues of g. Note that $\lambda(g^2) = 2\lambda(g)$.

Definition 1. We call g *loxodromic* if $\lambda(g) \in \mathfrak{a}_+^o \iff \lambda_1(g) > \dots > \lambda_d(g)$.

We would like to define a Zariski topology, it will be determined by the Zariski closed sets in $\mathrm{End}(V)$, which will be the set of zeros of a family of polynomials.

Example 1.

(1) $\mathrm{GL}(V)$ is Zariski open and Zariski dense in $\mathrm{End}(V)$.
(2) $\mathrm{SL}(V)$ is Zariski closed and Zariski connected in $\mathrm{End}(V)$.

Exercise 1. The Zariski closure H of a semigroup Γ is a group. *Hint*: Let $I^p = \{P \in \mathrm{Pol}(\mathrm{End}(V)) \mid P = 0 \text{ on } H, \deg P \leq p\}$. Set $(h \cdot P)(g) = P(gh)$. Check that $h \in H \iff h(I^p) \subset I^p$ for all $p \geq 1$.

Let $\Gamma \subset G$ be a Zariski dense semigroup, define

$$\Gamma_{\mathrm{lox}} = \{g \in \Gamma \mid g \text{ loxodromic}\}$$

we have a limit cone

$$L_\Gamma = \overline{\bigcup_{g \in \Gamma_{\mathrm{lox}}} \mathbb{R}_+ \lambda(g)} \subset \mathfrak{a}^+$$

and a group

$$\Delta_\Gamma = \overline{\langle \lambda(gh) - \lambda(g) - \lambda(h) \mid g, h, gh \in \Gamma_{\mathrm{lox}} \rangle} \subset \mathfrak{a}$$

Theorem 1 (Goldsheid-Margulis). *Γ_{lox} is Zariski dense in G.*

Theorem 2 (Benoist). *L_Γ is convex of nonempty interior.*

Theorem 3 (Benoist). *The group Δ_Γ is equal to $\mathfrak{a}$.*

The proofs of Theorems 5, 6 given here are due to Jean-Francois Quint.

1.2. **Loxodromic elements.**

Definition 2. We call $g \in G$ *proximal* if $\lambda_1(g) > \lambda_2(g)$.

If g is proximal, we can define $\pi_g = \lim_{n\to\infty} \frac{g^n}{\mathrm{tr}(g^n)} \in \mathrm{End}\,(V)$ is a rank-one projection.

Exercise 2. Let π be a rank-one projection and $g_n \in G$, $t_n \in \mathbb{R}$ such that $t_n g_n \xrightarrow[n\to\infty]{} \pi$ then g_n is proximal for $n \gg 0$.

Exercise 3. g is loxodromic if and only if for all i, $\bigwedge^i g$ is proximal in $\mathrm{End}\left(\bigwedge^i g\right)$. *Hint*: $\lambda_i(\bigwedge^i g) = \lambda_1(g) \cdots \lambda_i(g)$, $\lambda_2(\bigwedge^i g) = \lambda_1(g) \cdots \lambda_{i-1}(g)\lambda_{i+1}(g)$.

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Lemma 1. *For all i there exists $g \in \Gamma$ such that $\bigwedge^i g$ is proximal.*

Proof. Let $\pi \in \overline{\mathbb{R} \bigwedge^i \Gamma} \setminus 0 \subset \operatorname{End}\left(\bigwedge^i V\right)$ be of minimal rank r. We want $r = 1$. We can assume $\pi^2 \neq 0$. Let $W = \operatorname{Im} \pi$ then $\Delta = \pi \overline{\mathbb{R} \bigwedge^i \Gamma} \pi$ a semigroup and $\Delta \setminus 0 \subset \mathrm{GL}(W)$. Set $\Delta_1 = \Delta \cap \mathrm{SL}(W)$, note that Δ_1 is bounded.

Exercise 4. A compact semigroup in $\mathrm{SL}(W)$ is a group.

Hence there exists a basis of W such that $\Delta_1 \subset \mathrm{O}(\mathbb{R}^r)$ and $\Delta \subset \mathrm{Sim}(\mathbb{R}^r)$. Thus $\pi \overline{\mathbb{R} \bigwedge^i g_1 G} \pi \subset \mathrm{Sim}(\mathbb{R}^r)$. Since $\bigwedge^i G$ contains proximal elements, there is $\sigma \in \overline{\mathbb{R} \bigwedge^i G}$, a projection of rank one. Then $0 \neq \pi \bigwedge^i g_1 \sigma \bigwedge^i g_2 \pi \in \mathrm{Sim}(\mathbb{R}^r)$, which implies that $r = 1$. ☐

Proof of Theorem 1. For all$1 \leq i \leq d-1$, there exists $g_1 \in \Gamma$ such that $\bigwedge^i g_i$ is proximal. Let $S \subset \mathbb{N}$ be a subsequence such that for all i, j we define

$$\pi_{ij} = \lim_{n \in S} \frac{\bigwedge^i g_j^n}{\| \bigwedge^i g_j^n \|}$$

We know π_{ii} is rank one. Choose $h_1, \ldots, h_d \in \Gamma$ such that for all i

$$\tau_i = \bigwedge^i h_1 \pi_{i,1} \bigwedge^i h_2 \pi_{1,2} \bigwedge^i h_3 \cdots \pi_{i,d-1} \bigwedge^i h_d \in \operatorname{End}\left(\bigwedge^i V\right)$$

such that $\operatorname{tr}(\tau_i) \neq 0$. Notice that

$$\tau_i = \lim_{n \in S} t_n \bigwedge^i (h_1 g_1^n h_2 g_2^n \cdots h_d g_d^n h_d)$$

and $g = h_1 g_1^n h_2 g_2^n \cdots h_d g_d^n h_d$ loxodromic for $n \gg 0$, so Γ_{lox} is non-empty.

Choose $g \in \Gamma_{\text{lox}}$. Let $E = \{h_0 \in \Gamma \mid \operatorname{tr}\left(\bigwedge^i h_o \pi_{\bigwedge^i g}\right) \neq 0\}$ then for $n \geq 0$, $n_0 \gg 1$, $h_0 \in E$ we have $h_0 g^{n+n_0}$ is loxodromic and thus h_0 is in the Zariski closure of $\{h_- g^n, n \geq n_0\}$. So $E \subset \overline{\Gamma_{\text{lox}}}^{\text{Zar}}$ and Γ_{lox} is Zariski dense. ☐

1.3. **The limit cone.**

Definition 3. We call $g, h \in G$ *transversally proximal* if $\operatorname{tr}(\pi_g \pi_h) \neq 0$, and *transversally loxodromic* if for all i $\operatorname{tr}\left(\pi_{\bigwedge^i g} \pi_{\bigwedge^i h}\right) \neq 0$.

Lemma 2. *if g, h are transversally proximal, then for $n \gg 0$, $g^n h^n$ is proximal and*

$$\lim_{n \to \infty} \frac{\lambda_1(g^n h^n)}{\lambda_1(g^n) \lambda_1(h^n)} = |tr(\pi_g \pi_h)|$$

Proof. We have $\pm \frac{g^n}{\lambda_1(g^n)} \xrightarrow[n \to \infty]{} \pi_g$ it follows that $\pm \frac{g^n h^n}{\lambda_1(g^n)\lambda_1(h^n)} \xrightarrow[n \to \infty]{} \pi_g \pi_h$. But the right hand side is of rank 1 with $\operatorname{tr}(\pi_g \pi_h) \neq 0$, so $g^n h^n$ is proximal by Exercise 2. ☐

Corollary 1. *If g, h are transversally loxodromic, then for $n \gg 0$, $g^n h^n$ is loxodromic and $\nu(g, h) = \lim_{n \to \infty} \lambda(g^n h^n) - n(\lambda(g) + \lambda(h))$ exists in $\mathfrak{a}$. Furthermore, if $\nu(g, h) = (\nu_1, \ldots, \nu_d)$ then $\nu_1 + \cdots + \nu_i = \log \left| tr\left(\pi_{\bigwedge^i g} \pi_{\bigwedge^i h}\right) \right|$.*

Proof. Recall $\lambda(g) = (\log(\lambda_1(g)), \ldots, \log(\lambda_i(g))$ and $\lambda_1\left(\bigwedge^i g\right) = \lambda_1(g) \cdot \lambda_i(g)$. Applying Lemma 2 we get the convergence. ☐

Proof of convexity in Theorem 2. If $g, h \in \Gamma_{\text{lox}}$, then $\lambda(g) + \lambda(h) \in \Gamma_{\text{lox}}$ follows from Corollary 2. ☐

1.4. **The group Δ_Γ.** Note that $\nu(g,h) \in \Delta_\Gamma$.

Definition 4. g, h are *strongly transversally proximal* if $\mathrm{tr}(\tau_g \pi_h) \neq 0$, where τ_g is the projection on the sum of eigenspaces with $|\text{eigenvalue}| = \lambda_2(g)$.

Lemma 3. *Suppose g, h are strongly transversally proximal and fix $m \gg 0$. Then*

(1)
$$\lim_{n\to\infty} |tr(\pi_g \pi_{g^m h^n})| = \left| \frac{tr(\pi_g g^m \pi_h)}{tr(g^m \pi_h)} \right| = a_m(g,h)$$

(2)
$$\log(a_m(g,h)) \sim_{m\to\infty} c \frac{\lambda_2(g)^m}{\lambda_1(g)^m}$$

Proof.

(1) (**Exercise**) You can compute
$$\lim_{n\to\infty} \pi_{g^m h^n} = \frac{g^m \pi_h}{\mathrm{tr}(g^m \pi_h)}.$$

(2) Follows from
$$\log(a_m(g,h)) \simeq |a_m(g,h) - 1| = \simeq \frac{\mathrm{tr}((1-\pi_g) g^m \pi_h)}{\mathrm{tr}(g^m \pi_h)}.$$

□

Corollary 2. *Let g, h be strongly loxodromic then*

(1) *the limit $\alpha_m(g,h) = \lim_{n\to\infty} \nu(g, g^m h^n)$ exists in $\mathfrak{a}$.*
(2) *writing $\alpha_m(g,h) = (\alpha_{m,1}, \ldots, \alpha_{m,d})$ we have $\alpha_{m,1} + \ldots + \alpha_{m,1} \approx \frac{\lambda_{i+1}(g)^m}{\lambda_1(g)^m}$*

Proof of Theorem 3. Assume $\Delta_\Gamma \neq \mathfrak{a}$. Then there exists $\varphi \in \mathfrak{a}^* \setminus 0$ such that $\varphi(\Delta_\Gamma) \subset \mathbb{Z}$. Notice that $\alpha_m(g,h) \in \Delta_\Gamma$ then
$$\varphi(\alpha_m(g,h)) \sim \sum \varphi_i \left(\frac{\lambda_{i+1}(g)}{\lambda_i(g)} \right)^m.$$
Since Γ_{lox} is Zariski dense, choose $g \in \Gamma_{\mathrm{lox}}$ with different $\frac{\lambda_{i+1}(g)}{\lambda_i(g)}$ so that this is not an integer. □