EXPONENTIAL DECAY OF MATRIX COEFFICIENTS

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1. INTRODUCTION

We will look at the exponential decay of matrix coefficients of functions over $\Gamma\backslash G$ where G is a connected, simple, non-compact linear real group, $\Gamma < G$ a discrete subgroup and $\mathrm{d}x$ an invariant measure on $\Gamma\backslash G$. We want to ask the following questions,

Question 1. For $f_1, f_2 \in C_c(\Gamma\backslash G)$, to every g we can associate a correlation function: $g \mapsto \int_{\Gamma\backslash G} f_1(xg) f_2(x)\,\mathrm{d}x$.

(1) As $g \to \infty$ is there a limit for the correlation function?
(2) Is there a limit with exponential rate of convergence?
(3) is there a limit with *uniform* exponential rate of convergence for a given family $\{\Gamma_i < \Gamma\}$ of finite index subgroups?

Definition 1. A *unitary representation* of G is a group homomorphism $G \to U(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space, such that the map $G \times \mathcal{H} \to \mathcal{H}$, where $(g, v) \mapsto gv$, is continuous.

Definition 2. For $v, w \in \mathcal{H}$, the function $G \to \mathbb{C}$, $g \mapsto \langle gv, w\rangle$ is called the *matrix coefficient* with respect to v and w.

Consider $L^2(\Gamma\backslash G, \mathrm{d}x)$, here the inner product is given by $\langle f_1, f_2\rangle := \int_{x\in\Gamma\backslash G} f_1(x)\overline{f_2(x)}\,\mathrm{d}x$. This is Hilbert space and G acts on this space by right translation, $(g\cdot f)(x) = f(xg)$ and it preserves the inner product

$$\langle gf_1, gf_2\rangle = \int_{\Gamma\backslash G} f_1(xg)\overline{f_2(xg)}\,\mathrm{d}x = \langle f_1, f_2\rangle$$

so this gives us a unitary action of G. The matrix coefficient gives us exactly our correlation function. Thus any properties of a unitary representation and any statements about the matrix coefficient will also apply to the correlation function.

2. LIMIT OF THE CORRELATION FUNCTION

Theorem 1 (Howe-Moore '79)**.** *If ρ is a unitary representation of G with no G-invariant vector, then for all $v, w \in \rho$ (i.e. v, w are in the Hilbert space associated to ρ),*

$$\lim_{g\to\infty} \langle \rho(g)v, w\rangle = 0$$

Corollary 1. Let $f_1, f_2 \in L^2(\Gamma\backslash G)$, then

$$\lim_{g\to\infty} \int_{\Gamma\backslash G} f_1(xg) f_2(x)\,\mathrm{d}x = \begin{cases} \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \int f_1\,\mathrm{d}x \int f_2\,\mathrm{d}x & \text{if } \Gamma < G \text{ is a lattice} \\ 0 & \text{otherwise} \end{cases}$$

Proof. $L^2(\Gamma\backslash G)$ has no G-invariant vector if $\mathrm{vol}(\Gamma\backslash G) = \infty$. If $\Gamma < G$ is a lattice, then we can decompose $L^2(\Gamma\backslash G) = \mathbb{C} \oplus L^2_o(\Gamma\backslash G)$. □

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3. Exponential rate

Fix the Cartan decomposition $G = KA^+K$ where K is a maximal compact subgroup of G and A a maximal real-diagonalizable subgroup of G, A^+ is the positive Weyl chamber of this subgroup. A^+ is uniquely determined and can be written as $A^+ = \{\exp X \mid X \in \mathfrak{a}^+\}$.

Example 1. If $G = \mathrm{SL}(n, \mathbb{R})$, then $K = \mathrm{SO}(n)$, $A = \{\mathrm{diag}(e^{t_1}, \dots, e^{t_n}) \mid \sum t_i = 0\}$ and $A^+ = \{\mathrm{diag}(e^{t_1}, \dots, e^{t_n}) \mid t_1 \geq t_2 \geq \cdots \geq t_n, \sum t_i = 0\}$.

We define $\mathbb{R}\text{-rank}(G) = \dim A$. Then we consider the following cases, $\mathbb{R}\text{-rank}(G) \geq 2$, and $\mathbb{R}\text{-rank}(G) = 1$, which further breaks into $\mathrm{Sp}(n, 1)$ and F_4^{-20}, and $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$.

Definition 3. A non-compact subgroup $H < G$ is L^1*-tempered* if for every unitary represtntion of G, ρ with no invariant vector, and every K-fixed vectors v, w the matrix coefficient of v, w is in $L^1(H)$.

Theorem 2 (Margulis '97)**.** *Let $H < G$ be a closed and non-compact subgroup. Suppose H is an L^1-tempered subgroup, then G/H does not admit any compact quotient, that is, there is no discrete subgroup $\Gamma < G$ such that Γ acts properly discontinuously on G/H and $\Gamma \backslash H$ is compact.*

Remark 1. For this, it is not enough to show that there exists exponential decay we really want to prove very sharp exponential decay.

We define the function η_G, a bi-K-invariant function, such that $\eta_G(\exp x) = \frac{1}{2} \sum_{\alpha \in S} \alpha(x)$ for $x \in \mathfrak{a}^+$, where S is a maximal strongly orthogonal system of $\Phi_+(G, A)$, the positive root system. We call S *strongly orthogonal* if for distinct $\alpha, \beta \in S$, then $\alpha \pm \beta \notin \Phi$.

Theorem 3 (Oh, 2002)**.** *Assume $\mathbb{R}$-rank$(G) \geq 2$. For all $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that for any unitary representation ρ of G with no invariant vectors, and any v, w, K-finite unit vectors of ρ, then*

$$|\langle \rho(g) v, w \rangle| \leq C_\varepsilon \sqrt{\dim\langle Kv \rangle \dim\langle Kw \rangle} e^{-(1-\varepsilon)\eta_G(g)}$$

Example 2. If $G = \mathrm{SL}(n, \mathbb{R})$, we have, for a as in the previous example, $\Phi_+ = \{\alpha_{ij}(a) = t_i - t_j \mid i < j\}$. Then $S = \{\alpha_{i,n+1-i} \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ is a maximal strongly orthogonal system, and

$$\eta_G = \frac{1}{2} \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (t_i - t_{n+1-i})$$

In particular, if $n = 3$, we have

$$|\langle \rho(g) v, w \rangle| \leq C_\varepsilon (\dim\langle Kv \rangle \dim\langle Kw \rangle)^{1/2} e^{-(t_1 + \frac{t_2}{2})(1-\varepsilon)}$$

Since $S = \{t_1 - t_3\}$, and we can replace this with $2t_1 + t_2$.

Theorem 4 (Oh, 2002)**.** *If $G = SL(n, \mathbb{R})$, $n \geq 3$, or $Sp(2n, \mathbb{R})$, then this bound is optimal in "every" direction of g. That is, there exists an irreducible unitary representation ρ_0 of G and a K-fixed vector v_0 such that $C \cdot e^{-\eta_G(g)} \leq |\langle \rho_0(g) v, v \rangle|$ for all $g \in G$.*

3.1. **Where do we get the strongly orthogonal system.** To any $\alpha \in \phi$, we can associate a Lie group H_α, locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$, and generated by . In general the root space is not one dimensional, but we can generate a one dimensional subgroup from $\pm\alpha$. Look at the group

$$G_S := \langle H_\alpha, A \mid \alpha \in S \rangle$$

where S is strongly orthogonal. This group is reductive.

Proposition 1 (Main proposition)**.** *For any representation ρ with no G-invariant vector, if we consider $\rho|_{G_S}$ is a tempered, as defined by Harish-Chandra, representation of G_S.*

Proof of Thm 3. If we have a tempered representation, we understand its matrix coefficients completely, they are bounded by the Harish-Chandra function of S. □

Remark 2. Theorem 3 also holds for $G = \mathrm{Sp}(n, 1)$ and F_4^{-20} as follows from the classification of the spherical unitary dual (Kostant '69), but the bound we get is not optimal

Corollary 2. Suppose $\mathbb{R}$-rank of $G \geq 2$, or $G = \mathrm{Sp}(n,1)$ or F_4^{-20}. Let $f_1, f_2 \in C_c^\infty(\Gamma\backslash G)$, then

$$\langle gf_1, f_2\rangle = \begin{cases} \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \int f_1 \cdot \int f_2 + O(\|f_1\|_{\mathrm{sob}}\|f_2\|_{\mathrm{sob}}\rho^{-(1-\varepsilon)\eta_G(g)}) & \text{if } \Gamma < G \text{ is a lattice} \\ O(\|f_1\|_{\mathrm{sob}}\|f_2\|_{\mathrm{sob}}\rho^{-(1-\varepsilon)\eta_G(g)}) & \text{otherwise} \end{cases}$$

Remark 3. We have uniform exponential decay for any Γ.

3.2. **Rank 1.** If $G = \mathrm{SO}(n,1) = \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ or $G = \mathrm{SU}(n,1) = \mathrm{Isom}(\mathbb{H}^n_{\mathbb{C}})$, theorem 3 is not true. However, we can prove something for a very specific representation. Define

$$\rho_0 = \begin{cases} n-1 & \text{if } G = \mathrm{SO}(n,1) \\ 2n & \text{if } G = \mathrm{SU}(n,1) \end{cases},$$

let Δ denote the Laplacian on $\mathbb{H}^n$.

Theorem 5 (Lax-Phillips, Hamenstädt). *Consider $L^2(\Gamma\backslash\mathbb{H}^n)$, Γ a lattice. Then there exists only finitely many eigenvalues of $-\Delta$ on $L^2(\Gamma\backslash\mathbb{H}^n)$ in $[0, \rho_0^2/4)$. In particular, there is a spectral gap.*

We have

$$0 = \lambda_0 < \lambda_1(\Gamma) \leq \lambda_2(\Gamma) \leq \cdots \leq \lambda_m(\Gamma) < \frac{\rho_0^2}{4}$$

and we can write each eigenvalue as $\lambda_i(\Gamma) = s_i(\Gamma)(\rho_0 - s_i(\Gamma))$, where $\frac{\rho_0}{2} < S_i(\Gamma) \leq \rho_0$.

Corollary 3 (Shalom 2000). Let $\Gamma < G$ be a lattice, then if $f_1, f_2 \in C_c^\infty(\Gamma\backslash G)$, we have

$$\int_{X\in\gamma\backslash G} f_1(xa_t)f_2(x)\,\mathrm{d}x = \frac{\int f_1 \int f_2}{\mathrm{vol}(\Gamma\backslash G)} + O(\|f_1\|_{\mathrm{sob}}\|f_2\|_{\mathrm{sob}}e^{-(1-\varepsilon)(\rho_0 - s_1(\Gamma))t}$$

We cannot expect this uniform exponential error term for arbitrary finite index subgroups of Γ.

Example 3. Consider the case when $\Gamma \twoheadrightarrow \mathbb{Z}$, let $\Gamma_m = \ker(\Gamma \twoheadrightarrow \mathbb{Z}/m\mathbb{Z})$ and $\inf_m \lambda_1(\Gamma_m) = 0$

Definition 4. If there is a $\mathbb{Q}$-embedding $G \hookrightarrow \mathrm{SL}(N)$ and $\Gamma = G \cap \mathrm{SL}(N,\mathbb{Z})$, then Γ is called an *arithmetic lattice.*

Theorem 6 (Selberg, Burger-Sarnak, Clozel, Kelmer-Silberman[1]). *Let $G = SO(n,1)$ or $SU(n,1)$. Let Γ be an arithmetic lattice, then for all $q \in \mathbb{N}$, $\Gamma_1 = \{\gamma \in \Gamma | \gamma \equiv e \ mod(q)\}$, then* $\inf_q \lambda_1(\Gamma_q) > 0$.

Thus along these congruence subgroups we have uniform spectral gap and so Corollary 3 applies and we have uniform exponential mixing.

In the case when $G = \mathrm{SO}(2,1) = \mathrm{SL}(2,\mathbb{R})$ and $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, Selberg's $\frac{3}{16}$ theorem tells us that $\inf \lambda_1(\Gamma_q) \geq \frac{3}{16}$, but Selberg's eigenvalue conjecture would show that $\inf \lambda_1(\Gamma_q) \geq \frac{1}{4}$.

[1]At the end of the second lecture it is noted that Kelmer-Silberman must be included in the credit if we would like to make a statement about arithmetic lattices.