SEMIGROUPS IN SEMISIMPLE GROUPS

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1. Distribution of Eigenvalues

1.1. **Semisimple groups.** Let $V = \mathbb{R}^d$, $G = \mathrm{SL}(V) = \{g \in \mathrm{End}\,(V) \mid \det g = 1\}$ and define the Cartan subspace $\mathfrak{a} = \{x = (x_1, \dots, x_d) \mid x_1 + \cdots + x_d = 0\}$, inside we have the Weyl chamber $\mathfrak{a}_+ = \{x \in \mathfrak{a} \mid x_1 \geq \cdots \geq x_d\}$.

Definition 1. Let $\kappa : G \to \mathfrak{a}_+$ be the *Cartan projection*, satisfying $\kappa(g) = (\log \kappa_1(g), \dots, \log \kappa_d(g))$ where $\kappa_1(g) = \|g\| = \sup_{r \in V \setminus \{0\}} \frac{\|gv\|}{\|v\|}$ and $\kappa_i(g) = (i^{\text{th}}$ eigenvalue of $g^T g)^{1/2}$.

Exercise 1.

- Set $K = \mathrm{SO}(d)$ then we have the Polar decomposition $G = Ke^{\mathfrak{a}_+}K$. Show that $\kappa(g)$ is the unique element of $\mathfrak{a}_+$ such that $g \in Ke^{\kappa(g)}K$.
- $\| \bigwedge^i g\| = \kappa_1(g) \cdots \kappa_i(g)$.

Compare this to the Jordan projection we studied yesterday:

Definition 2. Let $\lambda : G \to \mathfrak{a}_+$ be the *Jordan projection*, satisfying $\lambda(g) = (\log \lambda_1(g), \dots, \lambda_d(g))$ where $\lambda_i(g) =$ modulus of the i^{th} eigenvalue of g.

Exercise 2.

- $\lambda(g) = \lim_{n\to\infty} \frac{1}{n} \kappa(g^n)$
- $\lambda_i(\bigwedge^i g) = \lambda_1(g) \cdots \lambda_i(g)$.

Define $F = \{\eta = (\eta_1, \dots, \eta_d) \mid \eta_i = i\text{-dimensional vector space}, \eta_1 \subset \cdots \subset \eta_i \subset \cdots \subset V\}$, a *flag variety.*

Definition 3. The *Iwasawa cocycle* is a map $\sigma : G \times F \to \mathfrak{a}$ where $\sigma(g, \eta) = (\log \sigma_i(g, \eta), \dots, \log \sigma_d(g, \eta))$ and $\sigma_1(g, \eta) = \frac{\|gv_1\|}{\|v_1\|}$, $\eta_1 = \mathbb{R}v_1$ and

$$\sigma_1(g_\eta) \cdots \sigma_i(g, \eta) = \frac{\left\| \bigwedge^i g(v_1 \wedge \cdots \wedge v_i) \right\|}{\|v_i \wedge \cdots \wedge v_i\|},$$

$\eta_i = \mathbb{R}v_1 \oplus \cdots \oplus \mathbb{R}v_i$.

Exercise 3. Show that σ is a cocyle, i.e. $\sigma(g_1 g_2, \eta) = \sigma(g_1, g_2\eta) + \sigma(g_2, \eta)$.

1.2. **Random walks on G.** Fix μ a probability measure on G, e.g. $\mu = \frac{1}{2}(\delta_a + \delta_b)$. Let $S = \operatorname{supp} \mu$, and assume S is compact. Define Γ to be the semigroup spanned by S, and assume Γ is Zariski-dense. We want to study the behavior of $\mu^{*n} = \mu * \cdots * \mu$, e.g.

$$\mu^{*n} = \frac{1}{2^n} \sum_{\substack{w = w(a,b) \\ \ell(w) = n}} \delta_w.$$

Specifically, we want to understand $\kappa_*(\mu^{*n})$.

We will now state the probabilistic statements of the main theorems

Let $C \subset \mathfrak{a}$ be a bounded convex subset. Fix $\eta \in F$.

Date: February 05, 2015.

Theorem 1 (Law of Large Numbers)**.** *The limit*

$$\lambda_\mu = \lim_{n\to\infty} \frac{1}{n} \int_G \kappa(g)\, d\mu^{*n}(g)$$

exists and $\lambda_\mu \in \mathring{\mathfrak{a}}_+$. *Moreover, if we assume* $0 \in \mathring{C}$, *then*

$$\mu^{*n}\left(\{g \in G \mid \kappa(g) \in n\lambda_\mu + nC\}\right) \xrightarrow[n\to\infty]{} 1$$

Theorem 2 (Central Limit Theorem)**.** *The limit*

$$\Phi_\mu = \lim_{n\to\infty} \frac{1}{n} \int_G (\kappa(g) - n\lambda_\mu)^{\otimes 2}\, d\mu^{*n}(g) \in S^2\mathfrak{a}$$

exists and is non-degenerate. Let N_ν *be the centered Gaussian law whose covariance 2-tensor is* Φ_μ. *Then*

$$\mu^{*n}\left(\{g \in G \mid \kappa(g) \in n\lambda_\mu + \sqrt{n}C\}\right) \xrightarrow[n\to\infty]{} N_\mu(C)$$

Theorem 3 (Local Limit Theorem)**.** *We have*

$$\sqrt{n}^{d-1} \mu^{*n}\left(\{g \in G \mid \kappa(g) \in n\lambda_\mu + C\}\right) \xrightarrow[n\to\infty]{} Leb_\mu(C)$$

where $Leb_\mu(C) = \lim_{p\to\infty} p^{d-1} N_\mu\left(\frac{C}{p}\right)$.

If you also fix $\eta \in F$ these three theorems are also true for $\kappa(g)$ replaced by $\sigma(g,\eta)$ and $\lambda(g)$.

Remark 1.

- LLT $\Rightarrow \Gamma_{\text{lox}} \neq \emptyset$ (Thm 1 from yesterday)
- CLT $\Rightarrow L_\Gamma$ has nonempty interior (Thm 2 from yesterday)
- LLT $\Rightarrow \Delta_\Gamma = \mathfrak{a}$ (Thm 3 from yesterday)

We will now look at how Thm 3 proves LLT.

1.3. **The transfer operator.** Fix a small $\gamma > 0$. Let

$$\mathcal{H}^\gamma = \left\{ \varphi \in C^0(F) \mid \sup_{\eta,\eta' \in F} \frac{|\varphi(\eta) - \varphi(\eta')|}{d(\eta,\eta')^\gamma} < \infty \right\}$$

and for every $\varphi \in C^0(F)$ we define the *transfer operator*

$$(P\varphi)(n) = \int_G \varphi(g\eta)\, \mathrm{d}\mu(g)$$

Then we have the following facts:

Fact 1.

- *There is a unique stationary probability a measure* ν *on* F, *i.e. for all* $\varphi \in C^0(F)$ *we have* $\nu(\varphi) = \nu(P\varphi)$.
- *Write* $\mathcal{H}^\gamma = \mathbb{C}1 \oplus \mathcal{H}^\gamma_0$ *where* $\mathcal{H}^\gamma_0 = \ker \nu$. *Then we get that spectral radius of* P *in* $\mathcal{H}^\gamma_0$ *is less than 1.*

Fix $\theta \in \mathfrak{a}^*$, we introduce the *complex transfer operator*, the analog of the Fourier transform,

$$(P_\theta\varphi)(\eta) = \int_G e^{i\theta(\sigma(g,\eta))} \varphi(g\eta)\, \mathrm{d}\mu(g)$$

Exercise 4. Show that the cocycle property gives

$$(P^n_\theta\varphi)(\eta) = \int_G e^{i\theta(\sigma(g,\eta))} \varphi(g\eta)\, \mathrm{d}\mu^{*n}(g)$$

Note that $\|P_\theta\|_\infty \leq 1$.

Fact 2. *For* $\theta \neq 0$ *the spectral radius of* P_θ *in* $\mathcal{H}^\gamma$ *is less than 1. Near* $\theta = 0$, *we have a* P_θ*-invariant decomposition* $\mathcal{H}^\gamma = \mathbb{C}\varphi_\theta \oplus \mathcal{H}^\gamma_\theta$. *Here* $P_\theta\varphi_\theta = \lambda_\theta\varphi_\theta$ *for* $\lambda_\theta \in \mathbb{C}$. *This is analytic in* θ.

Proof of Fact 2. The key point is the P_θ has no eigenvalue of modulus 1.

Assume $\theta \neq 0$ and $P_\theta \varphi = \mu\varphi$, for $\varphi \in \mathcal{H}^\gamma \subset C^0(F)$. Hence $|\varphi| \leq P(|\varphi|)$ and $\nu(|\varphi|) = \nu(P(|\varphi|))$. So φ has constant modulus on the support of ν. So, for all $g \in S$, and all $\eta \in \operatorname{supp}\nu$ we get

$$u\varphi(\eta) = e^{i\theta(\sigma(g,\eta))}\varphi(g\eta).$$

It then follows that for all $g \in S^n$ and all $\eta \in \operatorname{supp}\nu$

$$u^n\varphi(\eta) = e^{i\theta(\sigma(g,\eta))}\varphi(g\eta).$$

In particular, if $g \in S^n$ is loxodromic, and η_g^+ its attractive fixed point, then we have $\sigma(g, \eta_g^+) = \lambda(g)$ and so

$$u^n = e^{i\theta(\lambda(g))}.$$

Next, choose $g, h, gh \in \Gamma_{\text{lox}}$, then

$$e^{i\theta(\lambda(gh)-\lambda(g)-\lambda(h))} = 1$$

implies that $\theta(\Delta_\Gamma) \subset 2\pi\mathbb{Z}$. which contradicts Theorem 3. □

Proof of LLT. Define a quantity

$$(*) = \mu_{n,\eta}(C + n\lambda_\mu),$$

where $\mu_{n,\eta}$ is the image of μ^{*n} by $g \mapsto \sigma(g,\eta)$. We want to know if $(*) \approx \frac{\text{Leb}(C)}{\sqrt{n}^{d-1}}$. We consider the function $\psi = \mathbb{1}_C$, then

$$(*) = \int_{\mathfrak{a}^*} \hat{\mu}_{n,\eta}(\theta)\hat{\psi}(\theta)e^{-in\theta(\lambda_\mu)}\,\mathrm{d}\theta$$

by the Fourier Planchard formula. Write

$$\hat{\mu}_{n,\eta}(\theta) = \int_G e^{i\theta(\sigma(g,\eta))}\,\mathrm{d}\mu^{*n}(g)$$

We get that

$$\hat{\mu}_{n,\eta}(\theta) = (P_\theta^n 1)(\eta).$$

Now write $1 = a_\theta\varphi_\theta + \xi_\theta$ where $a \in \mathbb{C}$ and $\xi_\theta \in \mathcal{H}_\theta^\gamma$. Then

$$(*) \approx \int_{\mathfrak{a}^*} a_\theta\varphi_\theta(\eta)\lambda_\theta^n\hat{\psi}(\theta)e^{-in\theta(\lambda_\mu)}\,\mathrm{d}\theta.$$

Take an asymptotic expansion of $\lambda_\theta = e^{i\theta(\lambda_\mu)-\frac{1}{2}\varphi_\mu(\theta)} + O(\|\theta\|^3)$. To remove the error term, we need to use the same techniques as in the Abelian case. To finish we get

$$(*) \approx \int_{\mathfrak{a}^*} a_\theta\varphi_\theta(\eta)e^{\frac{n}{2}\varphi_\mu(\theta)}\hat{\psi}(\theta)\,\mathrm{d}\theta.$$

Taking $\theta = 0$ gives the conclusion. □