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Anosov representation and proper action

$G > H$ non-compact Lie group, G semi-simple

Q: Construct quotient manifolds $_{\Gamma}\backslash G/H$, Γ discret $< G$

$\Updownarrow$

$\Gamma \hookrightarrow G/H$ prop disc and free

Ex: ① $G/H = SO(1, n+1)/SO(1,n) = dS^{n+1}$
then Γ must be finite

② $G/H = SL_n(\mathbb{R})/SL_{n-1}(\mathbb{R})$ with n odd.
then Γ must be virt. cyclic.

2 general methods

"algebraic" (Kobayashi, Kulkarni, '80s) $\to \Gamma$ not Zariski dense.

ping pong in G/P (Benoist, '90 s) $\to \Gamma$ free group.

Goal: ① Use Anosov rep to construct proper actions of word hyperbolic group

② For $G/H = (G_0 \times G_0)/\mathrm{Diag}(G_0)$ where $\mathrm{rk}\, G_0 = 1$, $SO(1,n)$, $SU(1,n)$, $Sp(1,n)$

$= G_0$ as $(G_0 \times G_0)$-homogenous space: $(g_1, g_2)\, g = g_1 g g_2^{-1}$

all proper actions of quasi-isometry embedded groups come from this construction.

Thm 1: Γ finitely generated group.

Then $\{\rho \in \mathrm{Hom}(\Gamma, G_0 \times G_0) \mid \Gamma \underset{\rho}{\hookrightarrow} G_0$ prop. disc. q.i. embedding $\}$ ⊛ (e.g. quotient cpt)

open in $\mathrm{Hom}(\Gamma, G_0 \times G_0)$

$$⊛ = \{(\rho_L, \rho_R) \in \mathrm{Hom}(\Gamma, G_0)^2 \mid \rho_L \text{ convex cocompact } \underline{\text{and}}\ \rho_R \text{ ``uniformly shorter'' than } \rho_L\}.$$

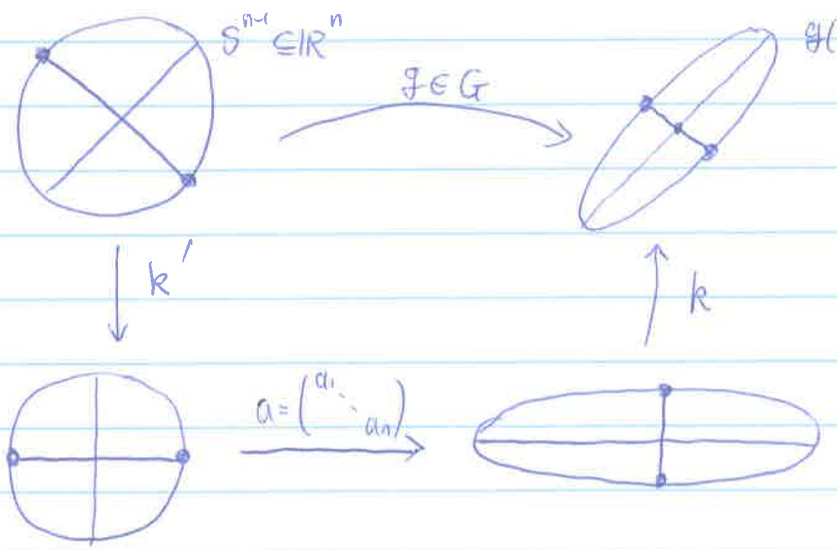
$$\text{shorter} \iff \sup_{\gamma \text{ of } \infty \text{ order}} \frac{\lambda(\rho_R(\gamma))}{\lambda(\rho_L(\gamma))} < 1.$$

$$\lambda : G_0 \to \mathbb{R}^+ \quad : \text{translation length in } \mathbb{H}^n_{\mathbb{K}}.$$
$$g \mapsto \inf_{p \in \mathbb{H}^n_{\mathbb{K}}} d(p, gp)$$

I Cartan decomposition

$$G = KA^+K \qquad \mu : G \to \log A^+ = \Omega^+ \ : \ \underline{\text{Cartan proj.}}$$
$$kak' \ni g \mapsto \log a$$

Ex: $G = SL_n(\mathbb{R})$, $K = SO(n)$, $A^+ = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in G \;\middle|\; \begin{array}{l} a_i > 0 \\ a_1 \geq \cdots \geq a_n \end{array} \right\}$



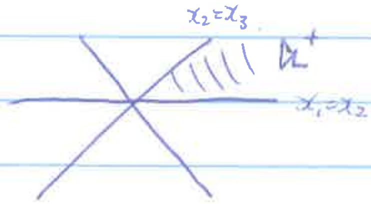
a_i = length of i-th ppal axis of $g(S^{n-1})$

$= \sqrt{i^{th} \text{ eigenvalue of } {}^t g g}$

$= i^{th}$ eigenvalue of g.

$\mathfrak{a}^+ = \{ x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0,\ x_1 \geq \cdots \geq x_n \}$.

$n=3$.



NB: $\| \mu(g_1 g g_2) - \mu(g) \| \leq \| \mu(g_1) \| + \| \mu(g_2) \|$

II Proper actions on G/H, H closed $< G$.

Properness criterion (Benoist, Kobayashi):

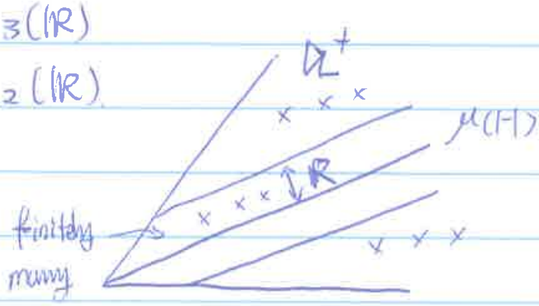
$\rho : \Gamma \to G$.

$\Gamma \underset{\rho}{\hookrightarrow} G/H$ prop. disc. $\iff$ $\mu(\rho(\Gamma))$ "goes away at infinity" from $\mu(H)$

i.e. $\forall R > 0$, $d(\mu(\rho(\gamma)), \mu(H)) \geq R$ for almost all $\gamma \in \Gamma$.

$G = SL_3(\mathbb{R})$

$H = SL_2(\mathbb{R})$.



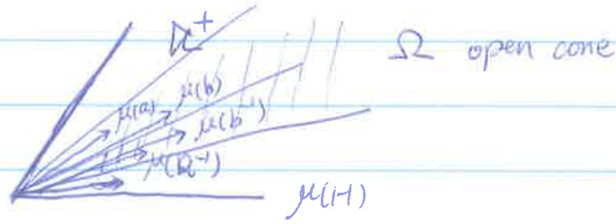
Proof of $\Leftarrow$: $\mathcal{C}$ cpt $\subset G$

$\rho(\gamma) \mathcal{C} H \cap \mathcal{C} H = \phi \iff \rho(\gamma) \in \mathcal{C} H \mathcal{C}^{-1}$

$\Rightarrow d(\mu(\rho(\gamma)), \mu(H)) \leq 2 \max_{\mathcal{C}} \| \mu \| < +\infty$.

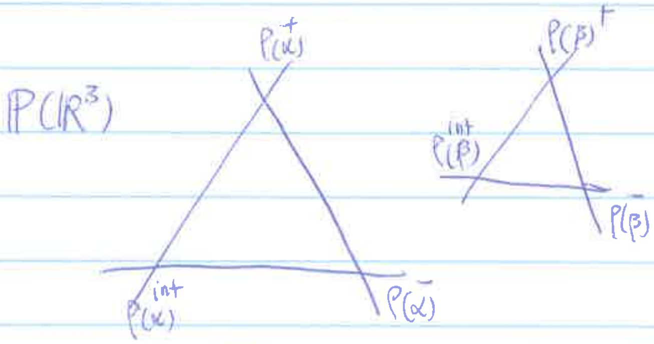
<u>Ex</u> (Benoist): $G = SL_3(\mathbb{R})$, $H = \left\{ \begin{pmatrix} t & & \\ & t & \\ & & t^{-2} \end{pmatrix} \,\middle|\, t > 0 \right\}$,

$\Gamma = \mathbb{F}_2 = \langle \alpha, \beta \rangle$.



<u>Claim</u>: $\exists\ \rho : \Gamma \to G$ inj and discrete with $\mu(\rho(\Gamma)) \subset \Omega$.

$\underset{\text{proper criterion}}{\Longrightarrow}$ $\Gamma \underset{\rho}{\hookrightarrow} G/H$ prop. disc.



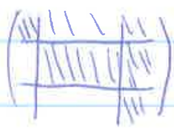
$\left.\begin{array}{l} \rho(\alpha) \sim a^n \\ \rho(\beta) \sim \beta^n \\ \quad n >> 1 \end{array}\right\}$ in generic position.

Γ play ping pang between them.

<u>III Anosov representation</u>

Γ word hyp group.

P parabolic subgroup of G (conjug. to its opposite).

<u>ex</u>: $G = SL_n(\mathbb{R})$ $P =$  $\to G/P = \{(p \in \text{proj hyp}) \subset \mathbb{P}(\mathbb{R}^n)\}$

<u>Def</u>: A rep $\rho : \Gamma \to G$ is P-Anosov if $\exists\ \xi : \partial_\infty \Gamma \to G/P$ ρ-equiv. which is
- continuous
- transverse (i.e. $\forall\ \eta \neq \eta'$ in $\partial_\infty \Gamma$, $\xi(\eta)$ and $\xi(\eta')$ are in generic position)

- dynamic-preserving $\left(\xi \begin{pmatrix} \text{attr fixed pt} \\ \text{of } \gamma \text{ in } \partial_\infty \Gamma \end{pmatrix} = \left\{ \begin{matrix} \text{attr fixed pt} \\ \text{of } \rho(\gamma) \text{ in } G/P \end{matrix} \right\} \right)$

  + "exponential contraction" property at $T_{\xi(\gamma)}(G/P)$

Thm 2. In the definition of P-Anosov, can replace exp contraction by

- $\ast$ $\mu(\rho(\Gamma))$ "goes away at ∞" from the walls of $\mathfrak{a}^+$ det by P
- or $\ast$ — — — — — linearly

  $\hookrightarrow d(\mu(\rho(\gamma)), \text{walls}) \geq C \|\mu(\rho(\gamma))\| - C'$

Cor: If $\rho: \Gamma \to G$ is P-Anosov, then $\Gamma \overset{\text{prop. disct.}}{\underset{\rho}{\hookrightarrow}} G/H$, $\forall H$ with $\mu(H)$ contained in union of walls det. by P

NB: $\{\text{parabolic subgp of } G\}/\text{conj} \overset{\sim}{\longleftrightarrow} \{\text{subsets of Walls}(\mathfrak{a}^+)\}$

min parab $\longleftrightarrow$ all walls

max parab $\longleftrightarrow$ one wall

Ex: $G = SL_n(\mathbb{R})$ $\quad \mathfrak{a}^+ = \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \middle| \begin{matrix} t_1 + \cdots + t_n = 0 \\ t_1 \geq \cdots \geq t_n \end{matrix} \right\}$

$\text{Wall}(\mathfrak{a}^+) = \{ \ker(e_i^* - e_{i+1}^*) \mid 1 \leq i \leq n-1 \}$.

$P = \begin{array}{l} i_1 \\ i_2 \\ \vdots \\ i_m \end{array} \left(\quad \right) \longleftrightarrow \{ \ker(e_i^* - e_{i+1}^*) \mid i \in \{i_1, \ldots, i_m\} \}$

Any Hitchin rep $\rho: \pi_1(S_g) \to SL_n(\mathbb{R})$ is $\left(\quad \right)$-Anosov

$\overset{\text{cor}}{\Longrightarrow} \pi_1(S_g) \underset{\rho}{\hookrightarrow} G/H$ e.g. for $H = \begin{cases} SL_m(\mathbb{R}),\ m < n-1 \text{ or } SO(p,q),\ |p-q| > 1 \\ SL_{\lfloor \frac{n}{2} \rfloor}(\mathbb{C}) \end{cases}$

IV Boundary map

Generalization construction:

$$g = k_g a_g k_g' \in KA^+K.$$

$$\xi : \partial_\infty \Gamma \longrightarrow G/P$$

$$\eta = \lim_n \gamma_n \longmapsto \lim_n k_{\gamma_n} P \in G/P$$

Thm 3. ξ is well defined, ρ-equivariant and continuous wherever $\sum_n e^{-d(\mu(\rho(\gamma_n)),\ \text{walls of } P)}$ CV uniformally over all geod $(\gamma_n) \in \Gamma^{\mathbb{N}}$ with $\gamma_0 = e$.

(e.g. if $\geq 2\log n - c$)

Idea of proof: $G = SL_3(\mathbb{R})$ $P = \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}$

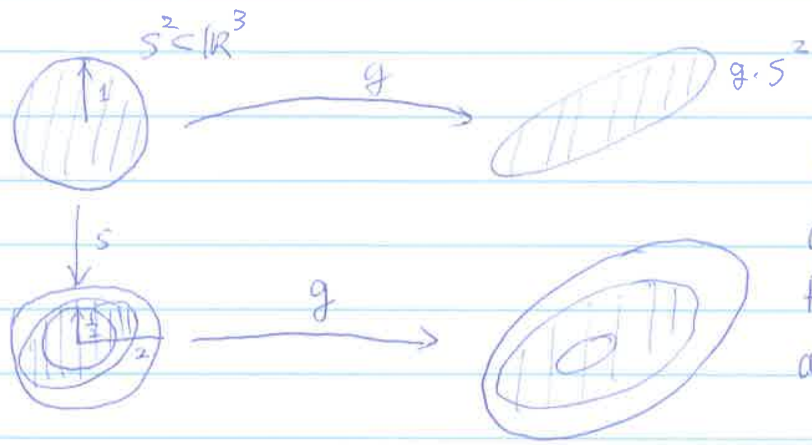
$$G/P = \{ (pt \in proj_{line}) \subset \mathbb{P}(\mathbb{R}^3) \}$$

$$eP \longmapsto (x, \ell)$$

Lemma S cpt $\subset G$. Then $\exists C > 0$ s.t. $\forall g \in G$, $\forall s \in S$.

$$d_{\mathbb{P}(\mathbb{R}^3)}(k_g \cdot x, k_{gs} \cdot x) \leq C e^{-d(\mu(g),\ 1^{st}\text{-wall})}$$

$\mathbb{P}(\mathbb{R}^{3*})$ 2^{nd} - wall



$k_g \cdot x$ is the direction of the 1^{st}-ppal axis of $g \cdot S^2$.

Apply to $S = \rho(\text{gen. set of } \Gamma)$

$g = \rho(\gamma_n)$.

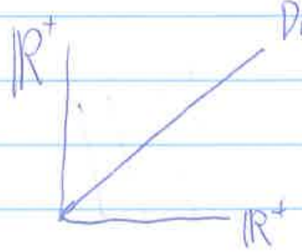
V $(G_0 \times G_0)/\mathrm{Diag}(G_0)$

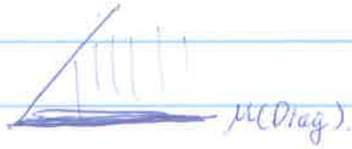
$G_0 = SO(1,n)$.

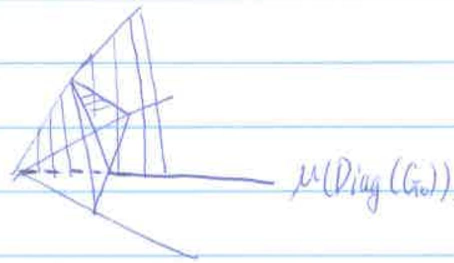
$G_0 \times G_0 \hookrightarrow SO(2,2n)$

$G_0 \times G_0 \hookrightarrow SO(n+1,n+1)$. $(n=2)$.

$\downarrow \mu$







Thm 4: $G_0 = SO(1,n)$, $SU(1,n)$, Sp. Γ f.g. gp. $\rho: \Gamma \to G_0 \times G_0$ QI embedding

TFAE: ① $\Gamma \overset{\rho}{\hookrightarrow} (G_0 \times G_0)/\mathrm{Diag}(G_0)$ prop disc.

② Γ word hyp and $\Gamma \overset{\rho}{\hookrightarrow} G_0 \times G_0 \hookrightarrow SO(2,2n)$ (SU, Sp)

is Anosov w.r.t. stab(isot. line).

③ ——— $SO(n+1,n+1)$ ———

④ ——— $\rho = (\rho_L, \rho_R)$ ρ_L convex cocompact

ρ_R "unif shorter" than ρ_L.