Christian Zickert **Coordinates for reps of 3-mfd groups** (J. w) S. Garoufalidis ~~S. Gar~~ D. Thursto

Let M be a compact, oriented 3-mfd with boundary.

$\{\rho: \pi_1(M) \longrightarrow SL(2,\mathbb{C})\}$ variety

$\{\rho: \pi_1(M) \longrightarrow SL(2,\mathbb{C})\}/_{conj}$ not variety

Restriction: Consider reps that are boundary-unipotent,

i.e. $\rho(\pi_1(\partial_i M)) \subseteq$ conjugate of $P = \{\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix} \mid * \in \mathbb{C}\}$

Reasons: – Typically, 0-dimensional (easier to compute)

– Well defined Chern-Simons, Bloch invariants.

– Volume is conjectrally a linear combination of hyp volumes form

Decorations: $B = \{\begin{pmatrix}x & a \\ 0 & x^{-1}\end{pmatrix} \mid \cdots \}$

Def. Let $\rho: \pi_1(M) \to SL(2,\mathbb{C})$ boundary-Unipotent.

A P-decoration (B-dec) of ρ:

is a ρ-equivariant map:

$$D: \hat{M}^{(0)} \longrightarrow SL(2,\mathbb{C})/P \quad (SL(2,\mathbb{C})/B)$$

i.e. an assygment of a coset to each boundary component of $\tilde{M}$

Motivation:

S = punctured surface $\chi(S) < 0$

Teichmüller space $\mathcal{T}(S) = \{$hyp structure$\}$

Decorated Teich space $\tilde{\mathcal{T}}(S) = \{$hyp structure + horocycles$\}$

$SL(2,\mathbb{R})/B \cong \partial \mathbb{H}^2$, $SL(2,\mathbb{C})/P \to$ horocycles

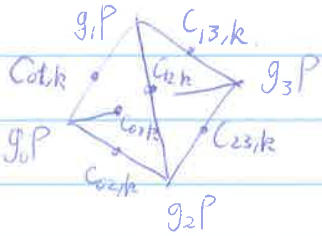
$gB \mapsto g\infty$

Fact: If ρ does not collapse a boundary component (i.e. $\rho(\pi_1(\partial M) \neq \{I\})$
then ρ boundary unipotent $\Rightarrow$ ρ has a B-decoration
ρ general $\Rightarrow$ ρ has 2^k B-decoration
$k = \#$ boundary components.

Ptolemy Coordinates:

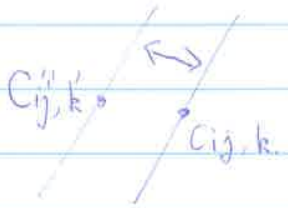
Let τ be an ideal triangulation of M.

Note: A decoration D asigns cosets to the vertices of each simplices



Def: D is generic if the cosets satisfies
$g_i e_1$ and $g_j e_1$ are independent.

Def. A Ptolemy assignment on τ is an assignment of a variable C_{ijk} to each oriented edge of each simplex. satisfying $C_{03,k}C_{12,k} + C_{01,k}C_{23,k} = C_{02,k}C_{13,k}$



then $C'_{i'j',k'} = C_{ij,k}$.

Thm (Garoufalidis - Thurston - Z)

There is a 1-1 correspondence

$$\{\text{Generic decoration}\} \xrightarrow{1:1} \{\underset{\overset{\parallel}{P(\tau)}}{\text{Ptolemy , assignment}}\} \xrightarrow{1:1} \{\text{natural cocycle}\}$$

$$\{g_0P, g_1P, g_2P, g_3P\} \mapsto C_{ij,k} = \det(g_i e_1, g_j e_1)$$

$$\alpha_{ij} = \begin{pmatrix} 0 & -C_{ij}^{-1} \\ C_{ij} & 0 \end{pmatrix}$$

$$\beta_{ij}^{k} = \begin{pmatrix} 1 & \frac{C_{ij}}{C_{ik}C_{kj}} \\ 0 & 1 \end{pmatrix}$$

Problem. The notion of "generic" depends on τ.

Thm (~~[illegible]~~ Goerner - Z)

There is a refined Ptolemy variety $\overline{P}(\tau)$

$\{\text{Decorations}\} \xrightarrow{1:1} \overline{P}(\tau) \longrightarrow \{\text{Bruhat cocycles}\}$

Also $\overline{P}(\tau) \xrightarrow{1:1} \overline{P}(\tau')$ regular isomorphism.

Rk: For P-decoration, $\mathbb{C}$ can be replaced by other field.

Generalizations:

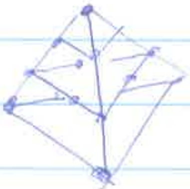
① $\pi_1(M) \to SL(n, \mathbb{C})$

$\Delta_n^3(\mathbb{Z}) = \{(t_0, t_1, t_2, t_3) \in \mathbb{Z}^4 \mid \Sigma t_i = n\}$

Def: A Ptolemy assignment on a simplex is a map

$$C : \Delta_n^3(\mathbb{Z}) \to \mathbb{C}^*$$
$$t \longmapsto C_t$$

s.t. for any $S \in \Delta_{n-2}^3(\mathbb{Z})$, have Ptolemy relation for vertices on same levels.



② Non - "boundary unipotent" reps

Thm (Z) There is an ~~[illegible]~~ enhanced Ptolemy variety $\mathcal{E}P(\tau)$ parametrize decorated $SL(2, \mathbb{C})$ - reps

Cor This compute the A - polynomial

Could be done for any Lie group ?