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Three proofs from dynamics of rigidity of surface group actions

I Geometricity $\Rightarrow$ Rigidity.

Def: Γ finitly generated $\circlearrowright X$ top space. is geometric action if given by

$$\underset{\text{cocompact lattice}}{\Gamma} \hookrightarrow \underset{\substack{\text{Lie group}\\ \text{transitive}}}{G} \hookrightarrow \mathrm{Homeo}(X).$$

Ex. $X = G/K$ e.g. symmetric space - - - - -

$X = S^1$:

$$\frac{G \subset \mathrm{Homeo}_+(S^1)}{S^1}$$

$PSL_2\mathbb{R}$ (2-fold cover)
$SL_2\mathbb{R}$
$\widetilde{PSL_2\mathbb{R}}^K$ (k-fold cover)

Γ cocompact.
Γ finite cyclic.
$\pi_1(\Sigma_g)$.
Virtually $\pi_1(\Sigma_g)$.

$$0 \to \mathbb{Z}/k\mathbb{Z} \to \widetilde{PSL_2\mathbb{R}}^K \to PSL_2\mathbb{R} \to 1$$

Thm (M.)

$\Gamma \xrightarrow{\rho_0} \mathrm{Homeo}_+(S^1)$ geometric action $\Rightarrow$ $\rho_0(\Gamma)$ is globally C^0-rigid

i.e. Any ρ in the same connected component of $\mathrm{Hom}(\Gamma, \mathrm{Homeo}_+ S^1)$ is semi-conjugate to ρ_0

$\exists\, h: S^1 \to S^1 \quad h\rho(\gamma) = \rho_0(\gamma)h \ \forall \gamma \in \Gamma$

Today: 3 Proofsketches

Focus on local rigidity

Easy case: - Γ finite cyclic.
- Not so bad to reduce virtually $\pi_1(\Sigma_g)$ to $\pi_1(\Sigma_g)$

Known: 1987 Matsumoto.
$\pi_1(\Sigma_g) \hookrightarrow PSL_2\mathbb{R}$ case. (maximal euler number).

Proof ①: Foliations (Ghys, Bowden)
~90 ~2013

Assume extra regularity.

$\rho: \pi_1 \Sigma_g \longrightarrow \mathrm{Hom}_+ S^1 \iff S^1 \to M^3 \downarrow \Sigma_g + \mathcal{F}$ foliation transverse to fibers.

$\rho_0: \pi_1 \Sigma_g \xrightarrow[\text{geom}]{} PSL_2\mathbb{R} \iff UT(\Sigma_g) + \mathcal{F}^u$ weak unstable for geod flow.

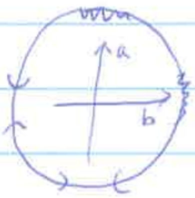
If ρ close to ρ_0, then $\mathcal{F}$ will be close to $\mathcal{F}^u$
$\Rightarrow \mathcal{F}$ transverse to $\mathcal{F}^u$
$\mathcal{F} \cap \mathcal{F}^u$ parametrize as ϕ^t close to geod flow

Structure stability (Anosov) $\Rightarrow \mathcal{F}$ equivalent to $\mathcal{F}^u$ & ρ conj to ρ_0

Problem: No hope for C^0

Proof ② Matsumoto (Arxiv Dec. 2014) "Ping - Pong"

Motivation: $\mathbb{F}_2 = \langle a, b \rangle$



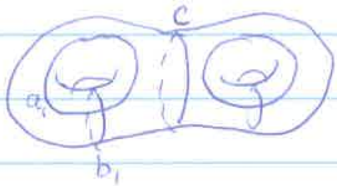
Combinatorial data

Stable under ~~particular~~ perturbation of a, b.

determines semi-conj.

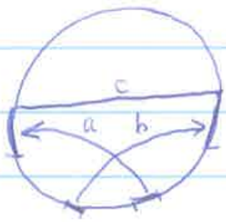
Works for $SL_2\mathbb{R}$, for $\widetilde{PSL}^K$

For $\pi_1 \Sigma_g$ $g = 2$



$\pi_1(\Sigma_g) = \underbrace{\mathbb{F}_2}_{\langle a,b \rangle} *_{\langle c \rangle} \mathbb{F}_2$

$PSL_2\mathbb{R}$, geometric case.



"ping pong"

Higher genus case.

— combination instruction.

Proof ③ (M –) "Work in coordinate"

$\Gamma \longrightarrow SL_2\mathbb{R}$ conj. class determined by $tr(\gamma)$

$\Gamma \longrightarrow Homeo_+(S^1)$ semi-conj class determined by $rot(\gamma)$

rot: $Homeo_+(S^1) \longrightarrow \mathbb{R}/\mathbb{Z}$

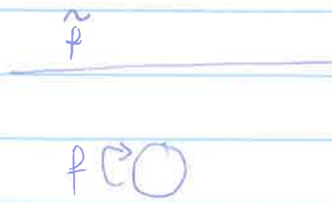
- continuous
- if $f \in Homeo_+(S^1)$ s.t. $f(x)=x$, then $rot(f)=0$.
- semi-conjugacy invariant. $hf_1 = f_2 h \Rightarrow rot(f_1) = rot(f_2)$

Thm (Calegari, Matsumoto) "Local coordinates"

$\forall \rho \in Hom(\Gamma, Homeo_+ S^1)$ $\exists$ nbhd $\mathcal{U}$ of ρ, so that any $\rho' \in \mathcal{U}$ is completely determined by $rot(\rho'(\gamma))$ (up to semi-conj) $\forall \gamma \in \Gamma$

Def (Poincaré)

$$rot(f) = \lim_{n\to\infty} \frac{\tilde{f}^n(x) - x}{n} \mod \mathbb{Z}$$



Upshot just need to prove rigidity of $rot(\rho(\gamma))$

$SL_2(\mathbb{R})$

$\pi_1(\Sigma_2)$

$rot([a_1, b_1]) = \frac{1}{2}$

Machinery combinatorial: Calegari – Walker.

II Rigidity $\Rightarrow$ Geometricity ?
Hyperbolicity

Conj: If $\rho_0 \in \mathrm{Hom}(\pi_1 \Sigma_g, \mathrm{Homeo}_+ S^1)$ is locally C^0 rigid
then ρ_0 is geometric.

Pf?

① Extra reg + foliation

Evidence: Thm (Morszczyk, 99)
"A nice foliation $\mathcal{F}$ on M^3 is infinitisemally rigid
iff $\mathcal{F}$ is transversely proj."

$\boxed{\text{has GV(J) class (cocycle) ?}}$

② Hands-on, combinatorial
rotation #'s ?