Ian Agol.

Entropy of pseudo-Anosov ~~which~~ with fix homology.
j.w. Christ Leininger and Dan Margalit.

Question: What are the shortest closed geodesics, in moduli space w.r.t. Teichmüller ~~iso~~metric (or various other metric?).

Equivalently, For a surface Σ_g, what is the minimal entropy of pseudo-Anosov map $\psi: \Sigma_g \to \Sigma_g$?

Let $L(g)$ denote the minimal such entropy.

Rk. $h(\psi) = \log(\lambda(\psi))$ $\lambda(\psi)$ is the dilatation.

Rk: $\dfrac{\log(2)}{3(2g-2)} \leq L(g) \leq \dfrac{\log(\varphi^+)}{2g-2}$

Penner, McMullen.

Aaber - Dunfield.
Hironaka
Kim - Takasawa

explicit example.

Rk. Let $L_{wp}(g)$ denote the systole length in the Weil-Petersson metric.

Thm (Brock - Bromberg) $\dfrac{C_1}{\sqrt{g-1}} \leq L_{wp}(g) \leq \dfrac{C_2}{\sqrt{g-1}}$

Kojima McShane.

Lower bound was discussed talk, using estimates coming from the renormalized volume by Kojima - McShane / Brock - Bromberg using idea of Schlenker.

$$\|\psi\|_{WP} \geq \frac{2}{3\sqrt{Area(S)}} Vol(N_\psi)$$

N_ψ = mapping torus = $\Sigma \times [0,1] / \{(x,0) \sim (\psi(x),1)\}$

upperbound.

Linch's inequality

$$\|\psi\|_T \geq \left(\sqrt{Area(S)}\right)^{-1} \|\psi\|_{WP}$$

What can be said about entropies of pseudo-Anosov's w/ various topological restrictions?

E.g. If ψ has orientable foliation

Lanneau-Thiffeault has found min dilatation / entropy pseudo-Anosov for genus 3, 4, 5.

Let $L_{\mathbb{F}}(k,g)$ = min entropy of a pseudo-Anosov map fixing a k-dim subspace of $H_1(\Sigma_g; \mathbb{F})$

Farb-Leininger-Margalit:

$$.197 \leq L_{\mathbb{Q}}(2g, g) \leq \log(62) \quad \text{(Torelli-Group)}$$

Ellenberg Conjectured:

$$\exists\, C_1, C_2 \quad \frac{C_1(k+1)}{g} \leq L_{\mathbb{Q}}(k,g) \leq \frac{C_2(k+1)}{g}$$

Upper bounds come from examples

Question on Ellenberg's blog. using Veering triangulations.

$$\frac{\log(k+2)}{Lg} \leq L_{\mathbb{Q}}(k,g)$$

Thm (A-Leininger-Margalit)

Let $\mathbb{F}$ be a field. For $g \geq 2$, $0 \leq k \leq 2g$.

$$0.00031 \left(\frac{k+1}{2g-2}\right) \leq L_{\mathbb{F}}(k,g) \leq 12 \log(2) \left(\frac{k+1}{2g-2}\right)$$

Rk: Only improves on Penner for $k > 372$.

Let $f: S \to S$ PA, $h(f)$ = entropy of f

$K_{\mathbb{F}}(f)$ = max dim fixed subspace of $H_1(S, \mathbb{F})$

Prop 2.1. Let S be a f° surface of finite type $f \in \mathrm{Mod}(S)$ pseudo-Anosov, then

$$0.00031 \left(\frac{K_{\mathbb{F}}(f)+1}{|\chi(S)|}\right) \leq h(f)$$

Prop 2.2. Let M^3 be complete orientable hyperbolic finite-volume then $b_1(M;\mathbb{F}) \leq 334.08 \, \mathrm{Vol}(M)$

Rk. Gelander proved the existence ~~daf~~ of some constant for each dimension, not explicit

Rk 334.08 is far from optimal

E.g. Thm (Culler-Shalen) $b_1(M;\mathbb{F}_2) \geq 6$

Then $\mathrm{Vol}(M) \geq 3.08$.

Proof of 2.2. Assume $b_1(M;\mathbb{F}) \geq 5$

Then $\log(3)$ is a Margulis constant for M.

M a hyp 3-mfd, ϵ-thick part $M_{\geq \epsilon}$ is a subset of M s.t. $\mathrm{inj}_x \geq \frac{\epsilon}{2}$

$\epsilon > 0$ the Margulis constant: $M_{\geq \epsilon}$ is the complement of solid tori & cusp of M.

<u>Rmk</u>. Makes use of results of Culler – Shalen "Paradoxical decompositions" + tameness.

<u>Jorgensen – Thurston argument</u>.

Let $V = \{v_1, \ldots, v_m\}$ be a max collection of pts in $M_{\geq \epsilon}$ s.t. $d(v_i, v_j) \geq \epsilon$, $i \neq j$ (ϵ-net).

Then $M_{\geq \epsilon} \subseteq \bigcup_{i=1}^{m} B_\epsilon(v_i)$.

$$\mathring{B}_{\epsilon/2}(v_i) \cap \mathring{B}_{\epsilon/2}(v_j) = \phi \quad i \neq j$$

$$\Rightarrow Vol(M) \geq m\, Vol(B_{\epsilon/2})$$

T_0 = graph $V(T_0) = V$.

$(v_i, v_j) \in E(T_0)$ if $d(v_i, v_j) \leq 2\epsilon$

(multiedges & loops).

$$b_1(T_0) \geq b_1(M)$$

$$\deg_{T_0}(v_i) \leqslant \frac{Vol(B_{5\epsilon/2}) - Vol(B_{\epsilon/2})}{Vol(B_{\epsilon/2})} = \nu$$

$$|E(T_0)| \leqslant \frac{m\nu}{2}.$$

$$\Rightarrow \underset{b_1(M)}{\overset{\vee\!\shortparallel}{b_1(T_0)}} \leqslant 1-(m-1)\frac{\nu m}{2} \leqslant 334.08\; Vol(M)$$

Thm (Kojima-McShane, Brock-Bromberg)

$$\mathrm{Vol}(N_f) \leq 3\sqrt{\tfrac{\pi}{2}|\chi(S)|}\ \|h\|_{WP} \leq 3\pi\,|\chi(S)|\,h(f)$$

$$b_1(M_f, \mathbb{F}) = K_{\mathbb{F}}(f) + 1$$

$$K_{\mathbb{F}}(f) + 1 \leq 334.08\ \mathrm{Vol}(M_f) \leq 334.08\ 3\pi\,|\chi(S)|\,h(f).$$

Cor 2.7 Let $K_{\mathbb{F}_2}(f) \geq 5 \longrightarrow h(f) \geq \frac{326}{|\chi(S)|}$