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(joint w. S. Choi)

Degeneration of convex real projective structure on open surface

Compactification.

Thurston: compactification of $\mathcal{T}(S) (\simeq \mathbb{R}^{6g-6})$

$\overline{\mathcal{T}}(S) \simeq \mathcal{T}(S) \cup \{\text{measured foliation}\}/\sim$

$$\begin{array}{c} \text{SI} \\ S^{6g-7} \end{array}$$

1. Construct a boundary
2. Geometric interpretation of boundary pts

$i : \mathcal{T}(S) \longrightarrow \mathbb{R}^{\mathcal{C}}_{\geq 0} \to \mathbb{P}(\mathbb{R}^{\mathcal{C}}_{\geq 0})$ $\quad \mathcal{C} = \{\text{s.c.c. on } S\}$

$h \longmapsto (\ell_h(\gamma))_{\gamma \in \mathcal{C}}$

If you prove that $i(\mathcal{T}(S))$ is ~~real~~ relatively compact, then $\partial i(\mathcal{T}(S))$ exists.

Morgen – Shalen

general construction

$V \subset \mathbb{R}^n$ connected components of a real alg. set $V_1 \subset \mathbb{R}^n$

$F = \{f_i\}_{i \in I}$ I countable $f_i \in \mathbb{R}[V_1]$

generating family

$i : V \longrightarrow \mathbb{R}^I_{\geq 0} \longrightarrow \mathbb{P}(\mathbb{R}^I_{\geq 0})$

$x \longrightarrow \log(|f_i(x)| + 2)$

Thm (M–S) 1) $i(V)$ is relatively compact

2) $j : V \longrightarrow (V \cup \{\infty\}) \times \mathbb{P}(\mathbb{R}^I_{\geq 0})$

$x \longmapsto (x, i(x))$

$\overline{V} = \overline{j(V)}$ $\quad j: V \to \overline{V}$ compactification.

3) $\partial V = \overline{V} \setminus V$ is a compact Hausdorff 2^{nd} countable topological space.

Γ f.g. group $\quad G = SL(n, \mathbb{R})$.

$V = X(\Gamma, SL(n, \mathbb{R}))$.

$F = (\{\rho \to tr\,\rho(\gamma)\})_{\gamma \in \Gamma}$

then $\overline{V} = \overline{X(\Gamma, SL(n, \mathbb{R}))}$

<u>Interpretation of boundary points:</u>

$G = SL(2, \mathbb{R})$ by M-S action of Γ on $\mathbb{R}$-trees

$G = SL(n, \mathbb{R})$ by Paulin, Parreau actions of Γ on $\mathbb{R}$-buildings.

(technic: asymptotic cones)

D. A.: you can also use the original M-S to get this result.

$SL(3, \mathbb{R})$. $\quad \Gamma = \pi_1(S)$

Cooper – Dehn Singular Hex metrics on S

<u>Generalisation of M-S thm:</u>

$V \subset \mathbb{R}^m$ real semi-algebraic set

$F = \{f_i\}_{i \in I}$ $\quad f_i : V \to \mathbb{R}_{>0}$

continuous, semi-algebraic

F is a <u>proper family</u> i.e. $\exists$ a finite subfamily $f_1, \dots, f_m$.

s.t. $V \to \mathbb{R}^m$ is a proper map.

$x \mapsto (f_1(x), \dots, f_m(x))$

$$i: V \longrightarrow \mathbb{R}^I \longrightarrow \mathbb{S}(\mathbb{R}^I) \qquad x \sim y \Leftrightarrow \exists \lambda > 0 \text{ s.t. } x = \lambda y$$
$$x \longmapsto (\log f_i(x))_{i \in I}$$

<u>Thm</u> (A) 1), 2) as above.

3) $\partial V = \bar{V} \setminus V$ is a compact Hausdorff 2^{nd} countable of topological dimension $\leq \dim V - 1$. $2\dim V - 1$

in particular it is homeomorphic to a compact subset of $\mathbb{R}$

4) if I is finite, then ∂V is homeomorphic to a polyhedral complex.

<u>Idea of the proof</u>:

4) related to the tropical geometry:

algebraic geometry $\longrightarrow$ polyhedral complex.

3). $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{F}$ subfamily

$\partial_{\mathcal{G}_1} V, \partial_{\mathcal{G}_2} V$

$\mathcal{G}_1 \subset \mathcal{G}_2 \qquad \partial_{\mathcal{G}_2} V \longrightarrow \partial_{\mathcal{G}_1} V$ subjective map.

inverse system.

$$\partial_{\mathcal{F}} V = \varprojlim_{\mathcal{G} \text{ finite.}} \partial_{\mathcal{G}} V \qquad \Rightarrow \text{boundary dimension.}$$

<u>Thm</u> (A) assume that $\exists \mathcal{G} \subset \mathcal{F}$ finite ~~fa~~ subfamily, every function ~~if~~ $\in \mathcal{F}$ is a rational function on the element of $\mathcal{G}$ with positive coefficient.

~~Thm~~ Then: $\partial_{\mathcal{F}} V \longrightarrow \partial_{\mathcal{G}} V$ is a homeo

in particular <u>PL</u>.

S closed surfaces $\quad Hit(3) \approx \mathbb{R}^{169-16}$

? $\partial Hit(3) \approx S^{169-17}$

(joint, S. Choi)

S open surface. $\Gamma = \pi_1(S)$ free group

$CDef(S) = \{$convex real proj structures on S with condition on bdy$\}/\sim$

every puncture must be: 1) Principle boundry component $\sim \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$ $\lambda_1 > \lambda_2 > \lambda_3$

2) quasi-principle holonomy $\begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix}$ $\lambda_1 \neq \lambda_2$

3) parabolic $\begin{pmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix}$

Prop: $CDef(S) \longrightarrow X(\Gamma, G)$ s.t. homeo with image which is a closed semi-alg set

Cor: $(\{\rho \to tr(\rho(\gamma))\})_{\gamma \in \Gamma}$ is a proper family
$\overset{\|}{F}$

$\partial_F CDef(S)$.

Use Fock-Goncharov coordinates

need to choose an invariant flag at every puncture

$CDef(S) = \{$proj str + flags$\}/\sim$

$CDef^+(S) \longrightarrow CDef(S)$ gen $6^{punt}:1$

$F \quad \partial_F(CDef^+(S)) = \partial_F CDef(S)$

$\mathcal{G} = \{$all F-G coordinates associates with all ideal triangulation$\}$

Prop: $\partial_{\mathcal{G}} CDef^+(S) \approx S^n$

fix 1-ideal triangulation, then all other coordinates are positive rational function on $\mathcal{G}$

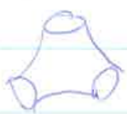
This $\partial_{\mathcal{G}} V$ is the Fock-Goncharov compactification.

$$\begin{array}{ccc} \mathcal{G} \cup \mathcal{F} & \partial_{\mathcal{G}\cup\mathcal{F}} V \simeq S^n & \\ \swarrow & & \searrow \\ S^n \simeq \partial_{\mathcal{G}} V & & V_{\mathcal{F}} V \end{array}$$

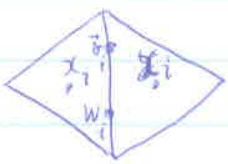
Thm (F-G)
all trace functions are positive Laurent polynomial on FG coordinates.

$\partial_{\mathcal{F}} V$ is the quotien of a sphere
Given $x \in S^n$.
$$C_x = \{ y \in S^n \mid x, y \text{ have the same image in } \partial_{\mathcal{F}} V \}$$
always a polyhedral complex.

$\exists$ exemple where collapsing actually happen



S^7



$CDef^+(S)$
$\partial_{\mathcal{G}} V \simeq S^n$.
If $x_i > 0$ & $w_i < 0$
& $z_i < -w_i$
then: no collapsing happens.