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joint with Brian Collier.

Asymptotics of certain families of Higgs bundles in Hitchin component.

$$\mathrm{Hom}(\pi_1(S), PSL(n,\mathbb{R}))/\!\!/PSL(n,\mathbb{R})$$

$$\cup$$

$$\mathrm{Hit}_n \underset{\text{Given } \Sigma}{\cong} \bigoplus_{j=2}^{n} H^0(\Sigma, K^{\otimes j}).$$

$$\ni q_2, \dots, q_n.$$

Parreau's compactification.

Higgs bundle.
a system of equations.

<u>Goal</u>: Understanding the asymptotic geometry of ρ_t in terms of asym of sub loci of $\bigoplus_{j=2}^{n} H^0(K^j)$.

Closely related to Pandit's talk

$(\overline{\mathcal{M}}_{DR} \leftrightarrow \overline{\mathcal{M}}_{Betti})$ $(\overline{\mathcal{M}}_{Higgs} \longleftrightarrow \overline{\mathcal{M}}_{Betti})$

this talk.

<u>For example</u>: $\rightsquigarrow$ a family of ρ_t-equiv. harm maps $f_t : \tilde{\Sigma} \to SL(n,\mathbb{R})/SO(n)$

$(f_t : \Sigma \longrightarrow \rho_t(\pi_1(\Sigma))\backslash SL(n,\mathbb{R})/SO(n))$.

As $t \to \infty$, $f_t(U)$ becomes more and more flat!

Fix Σ a Rieman surface

<u>Def</u>: A $SL(n,\mathbb{C})$-Higgs bundle over Σ is (Σ, ϕ) where

E — a rank n holo bundle over Σ with $\det E = \mathcal{O}$

ϕ ~~...~~ $\in H^0(\Sigma, \mathrm{End}(E)\otimes K)$ with $\mathrm{tr}\,\Phi = 0$

$$\phi : E \to E \otimes K$$

Def: A $SL(n,\mathbb{R})$- Higgs bundle is (E, ϕ, Q)
- (E,ϕ) is $SL(n,\mathbb{C})$- Higgs bundle
- $Q: E\times E \to \mathbb{R}$, nondegenerate.
- ϕ is Q symmetric i.e. $\phi^T Q = Q\phi$

Ex: $SL(2,\mathbb{R})$

Hopf diff from $\Sigma \to$ hyp surface

$$E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \quad \phi = \begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thm (rk 2 by Hitchin, general by Simpson).
Let (E,ϕ) be a stable Higgs bundle, then $\exists!$ Hermitian metric h s.t.

$$(*) \quad \underbrace{F_h}_{\text{curvature of Chern connection.}} + [\phi, \underbrace{\phi^{*h}}_{h^{-1}\bar{\Phi}^T h.}] = 0$$

Parametrization (Hitchin fiberation and section)

$$\begin{array}{c} \{SL(n,\mathbb{C})-(E,\phi)\} \\ s \uparrow \quad \downarrow \\ (P_2(\Phi), \ldots, P_n(\Phi)) \in \bigoplus_{j=2}^{n} H^0(\Sigma, K^j) \end{array}$$

$P_i(\Phi)$ is homogeneous $SL(n,\mathbb{C})$-Ad inv poly on $\mathfrak{sl}(n,\mathbb{C})$.

e.g.: $P_i(\Phi) = tr\Phi^i$

- $Im(S)$ is onto a component of $SL(n,\mathbb{R})$- Higgs bundles.

$S(q_2, q_3, \ldots, q_n) = (E, \phi, Q)$.

$$E = K^{\frac{n-1}{2}} \oplus \cdots \oplus K^{\frac{1-n}{2}} \qquad \phi = \begin{pmatrix} 0 & q_2 & \cdots & q_n \\ 1 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & q_2 \\ & & 1 & 0 \end{pmatrix} \quad Q = \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix}$$

$\underline{Q}$: Consider $(E, \Phi_t) \in \mathrm{Hit}_n$, $t \in \mathbb{R}^+ \to \infty$

Solving Hitchin eqn $\rightsquigarrow$ h_t on E

$\nabla_t = \nabla_{h_t} + \phi + \phi^{*h}$ flat connections on E

$T_\gamma(t) = E_{\gamma(0)} \longrightarrow E_{\gamma(L)}$ $\quad \gamma(s) \in \tilde{\Sigma}$ $\quad s \in [0, L]$

$\rightsquigarrow \rho_t : \pi_1(\Sigma) \to SL(n,\mathbb{R})$

$\nabla_t + h_t \longrightarrow \rho_t$ – equiv harm maps $f_t : \tilde{\Sigma} \longrightarrow SL(n,\mathbb{R})/SO(n)$.

– As $t \to \infty$, what is h_t, ∇_t, $T_\gamma(t)$, f_t ?

We restrict to • $\phi_t = \begin{pmatrix} 0 & 0 & & tq_n \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 & tq_{n-1} & 0 \\ 1 & \ddots & \ddots & tq_{n-1} \\ & \ddots & \ddots & 0 \\ 0 & & 1 & 0 \end{pmatrix}$

~~Thm (Collier–L)~~

Prop (Baraglia, Collier)
h is diagonal.

Thm (Collier, – L)

The solution h_t to $(*)$:

$$h_t = \mathrm{diag}\left(|tq_n|^{\frac{1-n}{n}}, |tq_n|^{\frac{3-n}{n}}, \ldots, |tq_n|^{\frac{n-1}{n}}\right)\left(1 + O(t^{-\frac{2}{n}})\right)$$

true for any nonzero pt ~~p~~ of q_n.

<u>Thm</u> (Collier – L) $\quad s \in [0, L]$

For any $|q_n|^{\frac{2}{n}}$-geodesic $\gamma(s) \subset \tilde{\Sigma}$ not passing through zeros of q_n

If γ "not heading close to zeros"

then

$$T_\gamma(t) = U \begin{pmatrix} e^{t^{\frac{1}{n}} \operatorname{Re}\int_\gamma q_n^{\frac{1}{n}}} & & & \\ & e^{t^{\frac{1}{n}} \operatorname{Re}\int_\gamma \xi_n q_n^{\frac{1}{n}}} & & \\ & & \ddots & \\ & & & e^{t^{\frac{1}{n}} \operatorname{Re}\int_\gamma \xi_n^{n-1} q_n^{\frac{1}{n}}} \end{pmatrix} U^{-1}\left(Id + O(t^{-\frac{1}{n}})\right)$$

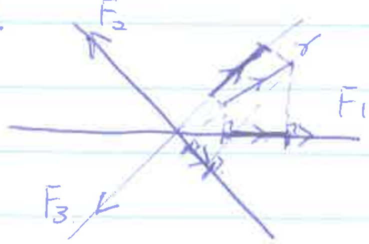
$\xi_n = e^{\frac{2\pi i}{n}}$

<u>Geometry</u>:

$$q_n = dz^n.$$

Locally, q_n defines n foliations with directional measure $\tilde{F}_1, \dots, \tilde{F}_n$

<u>$n=3$</u>



<u>Cor</u>: As $t \to \infty$, Hitchin eqn $F_{\nabla_h} + [\phi, \phi^{*h}] = 0$ decouples:

$$\begin{cases} F_{\nabla_h} = o(1) \\ [\phi, \phi^{*h}] = o(1) \end{cases}$$

i.e. ∇_{h_t} is going to be flat.

<u>$SL(2,\mathbb{C})$</u>: Taubes, Mazzeo – Swoboda – Weiss – Witt

<u>Rk</u>: $(0, q_3)$ case is proved by Loftin.

Our methods by Loftin's work and Dumas – Wolf.

Cor (harm map f_t).

$\forall p$ not zero of q_n, $\exists$ a nbhd U_p of p s.t.

$f_t(U_p)$ becomes more and more flat.

$\rightsquigarrow f_t : \tilde{\Sigma} \longrightarrow \left(SL(n,\mathbb{R})/SO(n), \frac{1}{t} d \right)$

$f_\omega : \tilde{\Sigma} \longrightarrow$ ~~Cone~~ Cone_ω

$f_\omega(U_p)$ is inside a 2-plane in an appartement of Cone_ω

Sketch of proof for h_t:

$E = K^{\frac{n-1}{2}} \oplus \cdots \oplus K^{\frac{1-n}{2}}$

$$\phi = \begin{pmatrix} 0 & & q_n \\ 1 & \ddots & \\ & 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} e^{\lambda^1} & & & \\ & e^{\lambda^2} & & \\ & & \ddots & \\ & & & e^{-\lambda^1} \end{pmatrix}$$

$$F_h + [\Phi, \Phi^{*h}] = 0.$$

$\uparrow$

$$\bar\partial(h^{-1}\partial h) \rightsquigarrow \begin{cases} \lambda^1_{z\bar z} + t^2 e^{-2\lambda^1} |q_n|^2 - e^{\lambda^1 - \lambda^2} = 0 \\ \lambda^2_{z\bar z} + e^{\lambda^1-\lambda^2} - e^{\lambda^2-\lambda^3} = 0 \\ \vdots \\ \lambda^{n/2}_{z\bar z} + e^{\lambda^{\frac{n}{2}-1} - \lambda^{\frac{n}{2}}} - e^{2\lambda^{\frac{n}{2}}} = 0. \end{cases}$$

Consider a base g s.t. $\begin{cases} g = |q_n|^{\frac{2}{n}} \text{ on } K \subset\subset \Sigma \\ \frac{|q_n|^{\frac{2}{n}}}{g} \leq 1 \text{ outside } K \end{cases}$

Define $u^j = \lambda^j - \frac{n+1-2j}{2} \ln g$, a function.

~~Then~~

then
$$\begin{cases} \cdots \\ \Delta u^3 = -e^{u^2-u^3} + e^{u^3-u^4} + \frac{n-5}{4} k_g. \\ \cdots \end{cases}$$

Maximum Principle. (Tool)

For example: at max of u^3:

$$0 \geq \Delta_g u^3 = -e^{u^2-u^3} + e^{u^3-u^4} + \frac{n-5}{4} k_g \quad (*)$$

$$\Rightarrow e^{u^2-u^3} \geq e^{u^3-u^4} + \frac{n-5}{4} k_g$$

$$\Rightarrow \frac{\max e^{u^2}}{\max e^{u^3}} \geq \frac{\max e^{u^3}}{\max e^{u^4}} - C.$$

(come from the sign in $(*)$)

Sketch of Proof for $T_\gamma(t)$

$$D = \nabla_h + \phi + \phi^{*h}$$

$$= \begin{pmatrix} \lambda^1_z & & tq_n \\ & \ddots & \\ & & \lambda^1_z \end{pmatrix} dz + \begin{pmatrix} 0 & e^{\lambda^1-\lambda^2} & & \\ & \ddots & \ddots & \\ & & \ddots & e^{\lambda^1-\lambda^2} \\ t\bar{q}_n e^{-2\lambda_1} & & & 0 \end{pmatrix} d\bar{z}$$

Restrict D on the path $\gamma(s) = se^{i\theta}$, $s \in [0, L]$.

Solving $T_\gamma(t) \Leftrightarrow D_{\gamma(s)} \Phi(s) = 0$.

$$\dot{\Phi}(s) = \left(t^{\frac{1}{\pi}} \begin{pmatrix} 0 & e^{-i\theta} & & e^{i\theta} \\ e^{i\theta} & \ddots & & \\ & & \ddots & e^{-i\theta} \\ e^{-i\theta} & & e^{i\theta} & 0 \end{pmatrix} + R \right) \Phi(s).$$