Geometric Nonlinear Dispersive PDE's 1

Daniel Tataru

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We start with an overview of the topic of geometric wave equations.

Consider the constant coefficient wave equation: $\Box u = 0 \in \mathbb{R}^{n+1}$.

In terms of notation, the wave equation can be associated with the Minkowski space-time. The Reimannian metric as a matrix (m^{ij}) can be written

$$(m^{ij}) = \begin{pmatrix} -1 & 0 & \dots & & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Utilizing the Minkowski metric we may formulate the wave equation as:

$$\partial_\alpha m^{\alpha\beta} \partial_\beta u = 0, \qquad u : \mathbb{R}^{n+1} \to \mathbb{R}.$$

By following the convention of raising and lowering indices, one may reformulate the wave equation:

$$\partial^\alpha \partial_\alpha u = 0$$

where the index was raised with respect to the Minkowski metric.

The variational formulation is

$$L(u) = \int \partial^\alpha u \cdot \partial_\alpha u = \int -u_t^2 + |\nabla_x u|^2$$

We can think of the wave equation as critical points of this Lagrangian.

Next we introduce a variety of problems.

Wave map

The simplest problem is the wave map equation. Start with a Riemannian manifold (M, g). Unlike for the wave equation, u will be mapped into the manifold:

$$u : \mathbb{R}^{n+1} \to M$$

The derivatives of u will be mapped into the tangent space of the manifold M:

$$\partial_\alpha u : \mathbb{R}^{n+1} \to T_u M$$

In this new setting we define the Lagragian of u:

$$\mathcal{L}(u) = \int_{\mathbb{R}^{n+1}} \langle \partial^\alpha u, \partial_\alpha u \rangle_g dxdt$$

One issue we face with wave maps is that it doesn't make a lot of sense to look at classes of solutions which are continuous. If we don't have uniform continuity, the global structure of M must be taken into consideration. When considering a continuous solution, locally the solution will stay within a small set within the target manifold; therefore, we could use local coordinates within the target manifold. If the problem is global, we cannot confine ourselves to the domain of a local chart on the manifold M, and therefore we need a more global approach.

We desire a way to write the wave map in a form invariant with respect to charts. This can be done as follows:

$$D^\alpha \partial_\alpha u = 0$$

where D^α is a notion of covariant differentiation.

One choice of coordinates (local chart) we can take:

$$\Box u_k = -\Gamma_k^{ij}(u)\partial^\alpha u_i \partial^\beta u_j, \quad u : \mathbb{R}^{n+1} \to \mathbb{R}^d$$

In local charts we lose track of global aspects so we want some global way of looking at the problem. We can think of (M, g) as isometrically embedded in $(\mathbb{R}^m, e)$ which follows from Nash's Theorem. If we do this, we get the following formulation of the WM equation:

$$\Box u_k = -S_k^{ij}(u)\partial^\alpha u \cdot \partial_\alpha u$$

The benefit of this new formulation is it has a global perspective rather than a local one.

Maxwell's Equation One invariant way of thinking of Maxwell's equation is by looking at a real valued connection A. If $u : \mathbb{R}^{n+1} \to \mathbb{C}$ then we may define derivatives:

$$D_\alpha u = (\partial + iA)u.$$

and curvature:

$$F_{ij} = \partial_i A_j - \partial_j A_i.$$

There is a natural Lagrangian:

$$\mathcal{L}(A) = \int F_{ij} F^{ij} dx$$

By looking at the Euler-Lagragian equation for this Lagrangian, we get Maxwell's Equation:

$$\partial^\alpha F_{\alpha\beta} = 0$$

There is a gauge invariance which means if

$$A \to A + db$$

where b is a scalar valued field then it remains a solution to the Maxwell equation.

Covariant wave equation: The Lagrangian is given by

$$\mathcal{L}(\phi) = \int D_A^\alpha \phi \cdot \overline{D_{A,\alpha}\phi} dx dt$$

Looking at the critical points of the Lagrangian reveals the covariant wave equation:

$$D^\alpha D_\alpha \phi = 0$$

Notice that both derivatives are covariant unlike in the wave map equation.

We couple the equations by adding their Lagrangians:

$$\mathcal{L}(\phi, A) = \int F_{ij}F^{ij} + D_A^{\alpha}\phi \cdot D_{A,\alpha}\phi$$

Note that the above equation depends on two variables so we will look separately for critical values with respect to ϕ and A. One will give the equation for ϕ and one will give the equation of A. When considering ϕ, note that F does not depend on ϕ and so we minimize the second term. This gives $\Box_A\phi = 0$, which is the covariant wave equation. Looking at critical points with respect to A yields $\partial_\alpha F_{\alpha\beta} = J \cdot \Im(\phi \cdot D_{A,\beta}\overline{\phi})$. Putting this together yields the Maxwell Klein Gordon equation (massless) which is given by

$$\begin{cases} \Box_A\phi = 0 \\ \partial^\alpha F_{\alpha\beta} = J := \Im(\phi \cdot D_{A,\beta}\overline{\phi}) \end{cases}$$

Since Maxwell's equation has some gauge invariance, we also get that the above MKG equation has some gauge invariance. The MKG gauge invariance:

$$(A, \phi) \to (A + db, e^{ib} \cdot \phi)$$

If we look from the PDE perspective and want to uniquely determine a solution then we need to find a way to uniquely determine a gauage invariance.

Yang Mills equation The third equation we will discuss is the Yang Mill equation.

G will be a lie group and g a lie algebra. Consider maps of the form

$$A : \mathbb{R}^{n+1} \to (g)^{n+1}$$

which is a lie algebra valued connection. If $B : \mathbb{R}^{n+1} \to g$ then the covariant derivative with the connection A is given by

$$D_A B := \partial_\alpha B + [A_{\alpha,B}]$$

The curvature is defined as

$$F_{\alpha\beta} = \partial^\alpha A - \partial^\beta A_\alpha + [A_\alpha, A_\beta]$$

The Lagrangian is given by (same as for Maxwell):

$$\mathcal{L}(A) = \int F_{\alpha\beta}F^{\alpha\beta}dxdt$$

Note that for this Lagrangian to make sense, we need an inner-product defined on our Lie algebra. This is provided by the killing-form inner product:

$$\langle A, B\rangle = \text{trace}(B^*, A)$$

This is a positive semi-definite inner product. The critical points of the Lagrangian solve the Yang-Mills equation:

$$D_A^\alpha F_{\alpha\beta} = 0$$

Lastly, we look at the gauge invariance. To talk about gauge invariance, we need to discuss the action of the Lie group on the Lie algebra. First a little notation:

$$Ad(\mathcal{O})B = \mathcal{O}B\mathcal{O}^{-1}, \quad \mathcal{O} : \mathbb{R}^{n+1} \to G$$

The gauge invariance is given by

$$A_\alpha \to \mathcal{O} A_\alpha \mathcal{O}^{-1} + \partial_\alpha \mathcal{O} \cdot \mathcal{O}^{-1}$$

Now we will talk about gauge fixing. We have two goals in mind when trying to fix the gauge:

1. Want to keep finite speed of propagation.
2. Want to keep the nonlinear structure as simple as possible.

There is no gauge that will optimally obtain both of these goals and so there is more than one gauge that we may consider.

For Maxwell's equation:

$$\partial^\alpha (D_\alpha A_\beta - \partial_\beta A_\alpha) = 0 \Rightarrow \partial^\alpha \partial_\alpha A_\beta = \partial_\beta \partial^\alpha A_\alpha$$

A reasonable gauge choice would be setting $\partial^\alpha A_\alpha = 0$ and so the above equation becomes $\partial^\alpha A_\alpha = 0$, the wave equation, and thus has finite speed of propagation. This is call the Lorenz gauge. This gauge will cause difficulties with the nonlinear problem, so let's consider a few other gauges as well.

The Coulomb gauge: $\partial_j A_j = 0$. This will split the space and the time components:

$$\begin{cases} \Box A_j = 0 \\ \Delta A_0 = 0 \end{cases}$$

The Laplace equation is nonlocal and so we have lost finite speed of propagation; however, this equation is easier to solve (compared to the Lorenz gauge) when nonlinearities are present.

The Temporal gauge: $A_0 = 0$. Then we get

$$\begin{cases} \Box A_j = 0 \\ \partial_0(\partial_j A_j) = 0 \end{cases}$$

This equation keeps finite speed of propagation and is decent for dealing with nonlinearities.

Another problem arises when we are looking at equations with nonlinearities. We may wonder how gauges differ for small/large data. It turns out that different gauges will be needed for different sizes.

Suppose we have Yang-Mills. Given data A we want (via gauge fixing) a normalized $\tilde{A}$. To do this, see if we can find some flow that will flow our map A into 0. Obviously 0 is associated with 0 as a normalized gauge. From here, do a pull-back from 0 to a normalized $\tilde{A}$. What should the flow be? Well the heat flow is one possible choice as it will flow things to 0.

Are there any gauge considerations for wave maps? There is a process of gauge fixing for the wave map equation. Consider the wave map equation $D^\alpha \partial_\alpha u = 0$ where $u : \mathbb{R}^{n+1} \to M$. The state space here is not a linear space. To address this issue look at flow of derivatives of u:

$$\partial_\beta u =: u_\beta$$

Applying the derivatives in the equation yields

$$D^{\alpha} D_{\alpha} u_{\beta} = R(u)(u^{\gamma}, u_{\gamma}) u_{\beta}$$

We get a curvature term that depends on our original map. Thus we have an issue unless our curvature is constant. To put this into coordinates all we need to do is pick an orthonormal frame in $T_u M$ (note that manifold doesn't need to be parallelizable). We obtain a gauge invariance:

$$u_{\alpha} \mapsto \mathcal{O} u_{\alpha} \mathcal{O}^{-1} + \partial_{\alpha} \mathcal{O} \mathcal{O}^{-1}$$