~~...~~ Arnaud Debussche 2

"Classical" SPDEs

Burgers with space-time infinite noise

$$du = (u_{xx} + (u^2)_x)\,dt + dW, \quad x \in (0,1)$$

Dirichlet: $u(0,t) = u(1,t) = 0$

W: cylindrical Wiener process on $H = L^2(0,1)$

Linear case: $dz = Az\,dt + dW$, $z(0)=0$

$A = \partial_{xx}$ + Dirichlet bc $\qquad z \in C_t^{1/4-} C_x^{1/2-}$

$V = u - z$

$$\begin{cases} \frac{dv}{dt} = Av + ((v+z)^2)_x \\ v(0) = u(0) = 0 \end{cases}$$

mild formulation: $v(t) = e^{At}u_0 + \int_0^t e^{A(t-s)} \partial_x[(v+z)^2]\,ds$

Fixed point: $C([0,T]; L^2)$ a.s.

$\mathcal{T}v(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}\partial_x[(v+z)^2]\,ds$

$v_1, v_2 \in C([0,T], L^2)$

$$(\mathcal{T}v_1 - \mathcal{T}v_2)(t) = \int_0^t e^{A(t-s)} \partial_x((v_1+z)^2 - (v_2+z)^2)\,dx$$

$$|\quad|_{L^2} \le \int_0^t \underbrace{|e^{A(t-s)}\partial_x|_{L^1\to L^2}}_{\le c|t-s|^{-3/4}} \underbrace{|v_1+v_2+2z|_{L^2}}_{\le 2(M+|z|_{L^\infty_{x,t}})} |v_1-v_2|_{L^2}\,ds$$

$$|(\mathcal{T}v_1 - \mathcal{T}v_2)|_{C([0,T],L^2)} \le c\int_0^t |t-s|^{-3/4}\,ds \cdot 2(M+|z|_{L^\infty_{x,t}})|v_1-v_2|_{C([0,T],L^2)}$$

Similar $\mathcal{T}B_m \to B_m$ for $T \le c(|u_0|_{L^2} + |v|_{L^\infty_{x,t}})^4$

$\to$ local existence

"Theorem" any deterministic eqn which is solved by contraction in a space $E \to$ space with additive noise ok if $z \in E$ a.s.

Global existence:

$$\frac{dv}{dt} = A v + \partial_x (v+z)^2$$

$\times v$ and $\int_0^1$

$$\to \frac{1}{2} \frac{d}{dt} |v|_{L^2}^2 + |v_x|_{L^2}^2 = \int_0^1 \partial_x \left((v+z)^2 \right) v \, dx$$

$$= - \int_0^1 (\cancel{v^2} + 2zv + z^2) \partial_x v \, dx$$

Cauchy Schwartz

$$\le 2 |z|_{L^\infty} |v|_{L^2} |\partial_x v|_{L^2} + |z|_{L^\infty}^2 |\partial_x v|_{L^2}$$

$$\frac{d}{dt} |v|_{L^2}^2 \le C \left(|z|_{L^\infty}^2 |v|_{L^2}^2 + |z|_{L^\infty}^4 \right) \to \text{gronwall}$$

- Fails for $du = (Au + (u^3)_x) dt + dW$
- The soln is adapted ("predictable")
- Similar in $L^p(0,1)$: $v_0 \in L^p \to u \in C([0,T], L^p)$ $p \ge 2$
  and in $C([0,1])$: $u_0 \in C[0,1] \to u \in C([0,T] \times [0,1])$.
- $\mathcal{L}(u(t, u_0))$ u_0 deterministic
  does not depend on ω

With this method we can solve

- Reaction diffusion

$$du = (\Delta u + u^3 - \lambda u) dt + dW$$

$d = 1 \to$ same method global existence

- 2D Navier-Stokes: $du = (\nu \Delta u + (u \cdot \nabla) u + \nabla p) dt + dW$

$$\operatorname{div} u = 0 \qquad \Phi$$

$\mathbb{P} = $ proj on $\{ u \in L^2, \operatorname{div} u = 0 \}$ $\quad A = \nu \mathbb{P} \Delta$

$$du = (Au + \mathbb{P}((u \cdot \nabla) u)) dt + dW$$

$$\text{2D} \quad dz = Az\,dt + \overset{\overset{\Phi}{\downarrow}}{} dw \quad z \in C([0,T], L^2)$$

$$z \in C([0,T]; L^4) \rightarrow \text{okay}$$

- If Burgers' $dW \rightarrow \Phi dW$

$$\Phi(-A)^{\beta} \in \mathcal{L}(H), \ \beta > 0$$

$\rightarrow$ easier

If $\mathrm{Tr}\,\Phi\Phi^* = |\Phi|^2_{\mathcal{L}_2(H)} < \infty$: global existence by Ito.

Ito formula: $H = \mathbb{R}$ $\quad dX = \lambda\,dt + b\beta$

$$d\varphi(x(t))\,?$$

$$\varphi(x(t+\delta t)) - \varphi(X(t)) = \varphi'(x(t))(\lambda\,dt + \overbrace{(\beta(t+\delta t) - \beta(t)}^{\sim \delta t^{1/2}})$$

$$+ \tfrac{1}{2}\varphi''(x(t))\underbrace{(\beta(t+\delta t) - \beta(t))^2}_{\sim \delta t}$$

$$d\varphi(x(t)) = \varphi'(x(t))\lambda\,dt + \varphi'(x(t))\,d\beta$$

$$+ \tfrac{1}{2}\varphi''(x(t))\,dt$$

Infinite dimensional: $du = (Au + (u^2)_x)\,dt + \Phi\,dW$

$$d\varphi(u) = (\nabla\varphi(u), Au + (u^2)_x) + (\nabla\varphi(u), \Phi\,dW)$$

$$+ \tfrac{1}{2}\mathrm{Tr}(\Phi\Phi^* D^2\varphi)$$

$$\varphi(u) = |u|^2_{L^2}, \quad D^2\varphi(u) = 2I$$

$$d|u|^2_{L^2} = 2(u, Au + (u^2)_x) + 2(u, \Phi\,dW) + \tfrac{2}{2}\mathrm{Tr}\,\Phi\Phi^*\,dt$$

$$|u|^2_{L^2} + 2\int_0^t |u_x|^2_{L^2}\,ds = |u_0|^2_{L^2} + 2\int_0^t (u, \Phi\,dW) + \mathrm{Tr}\,\Phi\Phi^*\,t$$

$$\mathbb{E}(|u|^2_{L^2}) + 2\mathbb{E}\int_0^t |u_x|^2_{L^2}\,ds = \mathbb{E}|u_0|^2_{L^2} + \mathrm{Tr}\,\Phi\Phi^*\,t$$

$$\mathbb{E}\left(\sup_{t\in[0,T]} |u|^2_{L^2}\right) \le \mathbb{E}\left(|u_0|^2_{L^2}\right) + 2\,\mathbb{E}\left(\sup_{[0,1]} \left|\int_0^t (u, \phi\, dW)\right|\right) + \operatorname{Tr}\phi\phi^* T.$$

$$\le 6\,\mathbb{E}\left(\int_0^T |\phi^* u|^2 ds\right)^{1/2}$$

Martingale inequality

5) Transition Semi group

$\forall u_0 \in H, \ \exists!\, u \in C([0,T], L^2)$ a.s. (Burgers)

$u(t, u_0)$ the soln

$$P_t \phi(u_0) = \mathbb{E}(\phi(u(t,u_0))) \qquad \phi \in \mathcal{B}_b(H)$$

Prove semigroup:

$$P_{t+s}\varphi(u_0) = \mathbb{E}(\varphi(u(t+s,u_0))) = \mathbb{E}(\phi(u(t+s, s; u(s,u_0))))$$

$$= \mathbb{E}\left(\mathbb{E}(\varphi(u(t+s, s; u(s,u_0)))\,|\,\mathcal{F}_s\right)$$

$$= \mathbb{E}(P_{t+s,s}\,\varphi\,(u(s,u_0)))$$

$$= P_s \underbrace{P_{t+s,s}\,\varphi}_{P_t\varphi}(u_0) \qquad \text{So semigroup}$$

$$P_t\phi(u_0) = \int_H \varphi(v)\, P_t(u_0, dv)$$

u_0 random law μ.

$$\mathbb{E}(\varphi(u(t,u_0))) = \mathbb{E}\left(\mathbb{E}(\phi(u(t,u_0))\,|\,\mathcal{F}_0\right)$$

$'' \int \varphi(v)\mu_t(v)$

μ_t law of $u(t,u_0)$ $\quad = \mathbb{E}(P_t\phi(u_0))$

$$= \int P_t\varphi(v)\,\mu(dv)$$

$$(\mu P_t) = P_t^*\mu = \mu_t$$

μ is inv if $P_t^* \mu = \mu$

$\mu_{t+r} = P_t^* \mu_r \qquad (\mu_r P_t)$

If " $\mu_t \to \mu$ " and P_t is Feller
$(\varphi \in C_b(H) \Rightarrow P_t \varphi \in C_b(H))$

$\rightarrow \mu$ is invarient.

Thm: (Krifler-Bogabrikov)
(P_t) Feller and $\frac{1}{t_n} \int_0^{t_n} P_s^* \mu \, ds \to \nu$ weakly
then ν is invarient.

Ex 1: Linear eqn

$$z(t) = \int_0^t e^{A(t-s)} \varphi \, dW(s)$$

$$\mathbb{E}(z) = 0, \quad \mathbb{E}((z,h)^2) \overset{\text{verience}}{=} \int_0^t |\varphi^* e^{A(t-s)} h|^2 ds$$
$$= \int_{-t}^0 |\varphi^* e^{As} h|^2 ds$$
$$\xrightarrow[t\to\infty]{} \int_{-\infty}^0 |\varphi^* e^{As} h|^2 ds$$

$$N\left(0, \int_{-\infty}^0 e^{As} \varphi \varphi^* e^{As} ds\right)$$
invarient

Burgers Smooth noise:

$$\mathbb{E} |u(t)|_{L^2}^2 + 2\int_0^t \mathbb{E}(|u_x(s)|_{L^2}^2) ds = \mathbb{E}(|u_0|_{L^2}^2) + t \, \mathrm{Tr} \, \varphi\varphi^*$$

$$\frac{1}{t} \mathbb{E} \int_0^t |u_x|_{L^2}^2 ds \le \frac{1}{2t} \mathbb{E}(|u_0|_{L^2}^2) + \mathrm{Tr} \, \varphi\varphi^*$$

$$\frac{1}{t} \int_0^t \mathbb{P}(u(s) \notin B_{H_0^1}(R)) ds \le \frac{1}{R^2} \left(\frac{1}{2t} \mathbb{E}(|u_0|_{L^2}^2) + \mathrm{Tr} \, \varphi\varphi^* \right)$$

$\longrightarrow \frac{1}{t} \int_0^t P_s^* \mu \, ds$ tight