Geometric Nonlinear Dispersive PDE's 2

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Recall the equations:

Wave Maps:

$$u : \mathbb{R}^{n+1} \to (M, g) \subset \mathbb{R}^m$$

$$\Box = -S(u) \cdot \partial^\alpha u \partial_\alpha u$$

Max-Klein-Gordon:

$$(\phi, A) : \mathbb{R}^{n+1} \to (\mathbb{C}, \mathbb{R}^{n+1}), \qquad \begin{cases} \Box_A \phi = 0 \\ \partial^\beta F_{\alpha\beta} = J_\alpha := \Im(\phi \overline{D}_\alpha \phi) \end{cases}$$

Recall that for the magnetic field, we have gauge freedom. If we fix the gauge, we would like to arrive at the wave equation for A rather than the curvature F (Recall $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$). With a suitable gauge, we may obtain

$$\begin{cases} \Box \phi = A \nabla \phi + A^2 \phi \\ \Box A = \phi \nabla \phi + A \phi^2 \end{cases}$$

Yang-Mills:

$$A : \mathbb{R}^{n+1} \to (g)^{n+1}, \qquad D_A^\alpha D_{A,\alpha} A = 0$$

After implementing the gauge freedom we arrive at the equation

$$\Box A = A \cdot \nabla A + A^3$$

Question: What are the symmetries of our problems? Below are two:

1. Scaling $\quad u(x,t) \to u(\lambda x, \lambda t).$
   $A(x,t) \to \lambda A(\lambda x, \lambda t)$
2. Lorenz invariance

Remark 0.1 $u(x,t) \to u(\lambda x, \lambda t)$ *is a natural scaling. For the wave map equation, u takes values in the manifold and so we cannot possibly have any scaling constant in front of u.*

We are interested in the solvability of this type of problem so we will look at initial value problems. Consider the wave map equation with initial data:

$$u(t=0) = u_0 \in H^2$$

$$u_t(t=0) = u_1 \in H^{s-1}$$

The simplest question we may ask is whether we have local solvability in $H^s \times H^{s-1}$.

The first test for the index s is to see to what extent the Sobolev space is consistent with the scaling. For each of these problems we can find a critical exponent s_c such that $H^{s_c} \times H^{s_c - 1}$ is invariant with respect to scaling. It can be show that the critical exponents are given by:

$$\begin{cases} s_c = \frac{n}{2} & \text{(WM)} \\ s_c = \frac{n}{2} - 1 & \text{(MKG and YM)} \end{cases}$$

$s > s_c$	$s = s_c$
local well-posedness	local = global
small data $\approx$ large data	small data $\not\approx$ large data
no global information	global information: scattering
semi-linear well-posedness	quasi-linear well-posedness
Picard/Contraction to solve	
$(u_0, u_1) \to u$ Lipschitz	$(u_0, u_1) \to u$ continuous

Remark 0.2 *For the $s > s_c$ case, even if we have large data on our initial time slice, we can chop the initial time slice into small pieces so that on each small piece we have small data. Then solve the wave equation on each small piece since we have finite speed of propagation. The downside to this is we don't obtain any global information. For the $s = s_c$ case, we are scaling invariant so the size of data does not change under scaling.*

Question: So what is Quasilinear well-posedness and how does it appear in the context of our equations?

We will use wave map notation, but everything we state also applies to Yang-Mill and MKG.

First, semi-linear well-posedness:
$S-$ space of solutions
$N-$ space for inhomogeneous term

$$\begin{cases} \Box u = \mathcal{N}(u) \\ u[0] = (u_0, u_1) \in H^s \times H^{s-1} \end{cases}$$

Above $\mathcal{N}(u)$ is our nonlinearity.

If we wish to solve this equation, we need a few pieces of information:

1. Information about the linear wave equation which will allow us to obtain an estimate $\|u\|_s \leq \|(u_0, u_1)\|_{H^2 \times H^{s-1}} + \|\Box u\|_N$.
2. To deal with the nonlinearity, we need mapping properties of the expression $\mathcal{N}$. For instance, show the map $u \to N(u)$ is a bounded map from S to N.
3. With these two properties, we can use Picard's iteration to solve.

Now for Quasilinear well-posedness:

$$\Box u_k = D\mathcal{N}(u_k) + f_k$$

u_k is the part of u that is at the frequency 2^k (see Littlewood-Paley theory for more information on the notation). The decomposition of u is $u = \sum_{k \in \mathbb{Z}} u_k$.

We would like to split the nonlinearity into a perturbative part and non-perturbative part.

$$\Box u_k = D\mathcal{N}(u_{<k})u_k + f_k$$

Here f_k is the perturbative part. We call $D\mathcal{N}(u_{<k})u_k$ the paradifferential part. What this means is we have high frequency wave driven by a local frequency background.

To solve this we need to show the following:

1. We want the map $u \to f_k$ to be perturbative and so it maps S into N.

2. We would like the estimate $\|u\|_S \leq \|(u_0, u_1)\|_{H^s \times H^{s-1}} + \|\Box^p u\|_N$

Here $\Box^p = \Box + D\mathcal{N}(u_{<k})P_k$.

This can't be solved with Fourier analysis and so parametrics are an alternative method. The methods are technical and briefly mentioned in the lecture.

Conclusions ($s = s_c$):

1. if (u_0, u_1) is small we hope to obtain the estimate $\|u\|_s \leq \|(u_0, u_1)\|_{H^s \times H^{s-1}}$.
2. Want to be able to say that more regular data implies more regular solutions. This can be shown with the estimate $\|u\|_{S^N} \leq \|(u_0, u_1)\|_{H^{s+N} \times H^{s-1+N}}$.
3. We also want a way of comparing solutions. To do this we look at the linearized equation:

$$\Box v = D\mathcal{N}(u)v.$$

   We can still get an estimate for our lineared equation if we lose some regularity. We call this weak Lipschitz dependence: $\|v\|_{S^{-\delta}} \leq \|(v_0, v_1)\|_{H^{s-\delta} \times H^{s-1-\delta}}$. This allows us to get uniqueness and obtain rough solutions as limits of smooth solutions; however, it's in a weaker topology.
4. To get convergence in a stronger topology we utilize frequency envelopes.

Let's look at frequency envelops. First define $a_k := \|(u_{0k}, u_{1k})\|_{H^s \times H^{s-1}}$. By Plancherel's theorem, $\{a_k\} \in \ell^2$.

Now at each k we have some amount of energy. We would like to be able to say that when we look at the solution, the distribution of energy for the solutions will follow a similar pattern. i.e. if we have all our energy at one frequency initially, perhaps all the energy won't go to another frequency.

Leakage of energy to other frequencies has to have some decay. Thus if our initial distribution of energy is wildly varying, we can cover it with something that isn't wildly varying. i.e. take $c_k \geq a_k$ where c_k are slowly varying and $\left|\frac{c_k}{c_j}\right| \leq 2^{\delta|j-k|}$. Once we have this nicely varying cover, if our initial data is under a frequency envelop, then the solution remains under the frequency envelop. To write this as an estimate:

$$\|u\|_{s_c} \leq \|(u_0, u_1)\|_{(H^s \times H^{s-1})_c}$$

Some results:

- (WM) globally well-posed for small-data in $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}$ where $d \geq 2$. Tau ($\mathbb{S}^2$), Krieger ($\mathbb{H}^2$), Tataru (general).
- (MKG) Globally well-posed for small data in $\dot{H}^{\frac{d}{2}-1} \times \dot{H}^{\frac{d}{2}-2}$ for $d \geq 4$. (Krieger, Stubert, Tataru)
- (YM) Globally well-posed for small data in $\dot{H}^{\frac{d}{2}-1} \times \dot{H}^{\frac{d}{2}-2}$ for $d \geq 4$. (Krieger, Tataru)

All of these equations have a conserved energy: $E = \frac{1}{2}\int \|u_t\|_g^2 + \|\nabla_x u\|_g^2 dxdt$.

The energy-critical dimension is: $n = 2$ for (WM) and $n = 4$ for (MKG,YM).