Jonathan Mattingly Lecture 2

Recall $\mathcal{X}$, P $(*)$ $\inf\limits_{x \in \mathcal{X}} P(x, \cdot) \geq \alpha \nu(\cdot)$

$\nwarrow$ prob measure

Thm: If $(*)$ holds then

$$\| \mu_1 P^n - \mu_2 P^n \|_{TV} \leq (1-\alpha)^n \| \mu_1 - \mu_2 \|_{TV}$$

Where $\| \nu_1 - \nu_2 \|_{TV} = \frac{1}{2} \sup\limits_{|\varphi|_\infty \leq 1} \int \varphi(u) \nu_1(du) - \int \varphi(u) \nu_2(du)$

$$= \sup_{A \in \Sigma} | \nu_1(A) - \nu_2(A) |$$

How the proof works:

Create a new transition kernel: $\tilde{P}(x, \cdot) = \dfrac{P(x,\cdot) - \alpha \nu(\cdot)}{1-\alpha}$

Start chain at (X_0, Y_0). Aux. var. $\tilde{X}_n \sim \tilde{P}(X_{n-1}, \cdot)$

~~At each step~~

$\tilde{Y}_n \sim \tilde{P}(Y_{n-1}, \cdot)$

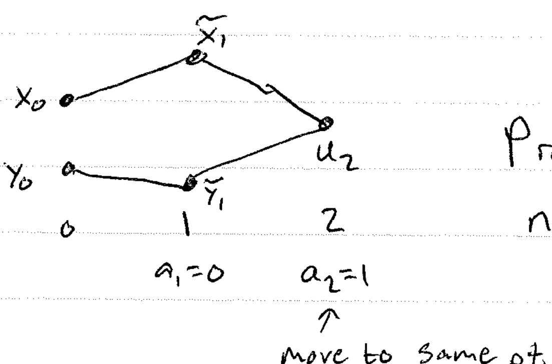
$U_n \sim \nu(\cdot)$ iid

$a_n \sim \mathrm{Ber}(\alpha)$ iid

$\hookrightarrow$ bernoulli: (coin flip)

$X_n = a_n U_n + (1-a_n) \tilde{X}_n$

$Y_n = a_n U_n + (1-a_n) \tilde{Y}_n$ if $X_{n-1} \neq Y_{n-1}$ otherwise $Y_n = X_n$



Prob of never couple after n steps: $(1-\alpha)^n \longrightarrow 0$

$\square$

Note: we could replace $(*)$ with

$$(**) \quad \inf_{x,y \in \mathcal{X}} \| P(x,\cdot) - P(y, \cdot) \|_{TV} \leq 1 - \alpha$$

Recall $\| \nu_1 - \nu_2 \| = 1$ iff $\nu_1 \perp \nu_2$

Ex: $du(x,t) = \Delta u(x,t)\,dt + \sum_{k=1}^{\infty} \sigma_k \sin(2\pi x k)\,dB_t^{(k)}$

$x \in [0,1]$ $\quad u(0,t) = u(1,t) = 0 \quad \sigma_k > 0, \{\beta_t^{(k)} : k \in \mathbb{N}\}$

iid std BM.

then

$$u(x,t) = \sum_k u_k(t) \sin(2\pi x k)$$

$$du_k(t) = -(2\pi k)^2 u_k(t)\,dt + \sigma_k\, dB_t^{(k)}$$

$$u_k(t) \sim N(m_k(t), S_k(t))$$

$$m_k(t) = e^{-(2\pi k)^2 t} u_k(0)$$

$$S_k^2(t) = \frac{\sigma_k^2}{2(2\pi k)^2}\left(1 - e^{-2(2\pi k)^2 t}\right)$$

We can explicitly write the transition Kernel:

$$P_t\left(u_0, \prod_{k=1}^{\infty} dx_k\right) = \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi S_k^2(t)}} e^{-\frac{(x_k - m_k(t))^2}{2 S_k^2}} dx_k$$

Lemma: $P_t(u_0, \cdot) \sim P_t(v_0, \cdot)$ iff

$$\sum \frac{|k|^2}{\sigma_k^2} |\hat{u}_k(0) - \hat{v}_k(0)|^2 e^{-8\pi^2 |k|^2} < \infty$$

and singular otherwise.

If $u_0, v_0 \in L^2([0,1])$ then transition measures are equivalent if $\sigma_k = \frac{1}{|k|^p}$ $\quad p > 0$

But if $\sigma_k = e^{-c|k|^3}$ then $\exists\, u_0$ and v_0 st $P_t(u_0, \cdot) \perp P_t(v_0, \cdot)$ $\forall t \geq 0$.

Initial conditions u_0, v_0 $\quad \rho(t) = u(t) - v(t)$
with same BM.

$$\partial\rho/dt = \Delta\rho \quad \to \quad \frac{d}{dt}\|\rho\|_{L^2_x} \le -\|\nabla\rho\|^2_{L^2_x} \le -\lambda_1\|\rho\|_{L^2_x}$$

$$\frac{1}{T}\int_0^T \phi(u(t))\,dt \xrightarrow{a.s.} \int\phi\, d\mu_1, \qquad \frac{1}{T}\int_0^T \phi(v(t))\,dt \xrightarrow{a.s.} \int\phi\, d\mu_2$$

$u_0 \sim \mu_1 \qquad\qquad v_0 \sim \mu_2$

must be equal by previous line.

$$\|\nu_1 - \nu_2\|_{W_1} = \sup_{\{\phi : lip(\phi) \le 1\}} \int \phi(u)\, d\nu_1(u) - \int \phi(u)\, d\nu_2(u)$$

$$du_t = -Lu_t\,dt + N(u_t)\,dt + \Phi\, dW_t$$

$\Phi\, dW_t \hookrightarrow \sum g_k\, dW^{(k)}$ — to simplify say it's $\nabla_k f_k$

L: pos. def. linear operator with cpt resolvent

g_k: functions

assume $\langle N(u) - N(v), u - v\rangle \le K_1 \|u - v\|^2$ $\qquad \|\rho\|_s = \sum_k \lambda_k \langle \rho, f_k\rangle^2$

eigen basis $Lf_k = \lambda_k f_k$

$$\rho_t = u_t - v_t \qquad \partial/\partial t = -L\rho_t + N(u_t) - N(v_t)$$

$$\frac{d}{dt}\|\rho_t\|_0^2 = -2\|\rho_t\|_1^2 + 2\langle N(u_t) - N(v_t), u_t - v_t\rangle$$

$$\le -2\lambda_1\|\rho_t\|_0^2 + 2K_1\|\rho_t\|^2$$

$$= -2(\lambda_1 - K_1)\|\rho_t\|_0^2$$

If $\lambda_1 < K_1$ then just like in previous example.

$\exists!$ inv measure with exp conv in W_1-Metric.

$$\Pi_n u = \sum_{k=1}^{n} f_k \langle u, f_k \rangle$$

What if we are in the other setting ie $\lambda_1 \le K_1$?

Thm: Assume $\exists n$ so that $\| \Pi_n^{\perp} [N(u) - N(v)] \| \le K_2 \|u-v\|^2$ with $K_2 < \lambda_n$ and range $(\Phi) \supset$ range (Π_n), then $\exists!$ inv measure with exp conv in the W_1-metric.

$u_t(u_0, W)$ ↑ Brownian path , $v_t(v_0, \tilde{W}_t)$ ↳ not Brownian path.

$$\tilde{W}_t = W_t + \int_0^t \dot{H}_s \mathbb{1}_{A_s}(w)\, ds$$

$$A_s = \{ w = \int_0^1 \| \dot{H}_s(w) \|^2 ds \le R \} \subset \Omega$$

$$\dot{H}_s = \Phi^{-1}\left(L r_t - \Pi_n [N(u_t) - N(v_t)] - \frac{r_t}{\|r_t\|} \right) \quad \begin{cases} \rho_t = \Pi_n^{\perp}(u_t - v_t) \\ r_t = \Pi_n (u_t - v_t) \end{cases}$$

<u>assume we stay in A_s</u>

$$\frac{\partial}{\partial t} r_t = -\frac{r_t}{\|r_t\|} \qquad \partial_t \rho = -L \rho_t + \Pi_n^{\perp} [N(u_t) - N(v_t)]$$

$$\frac{d}{dt} \| r_t \|^2 = -2 \frac{\|r_t\|^2}{\|r_t\|} = -2 \| r_t \|$$

---

if μ and ν are ergodic inv measures then TFAE

1) $\mu = \nu$
2) $\exists$ an asymptotic coupling $u(t)$ w/ $u(0) \sim \mu$, $v(t)$ w/ $v(0) \sim \nu$, $\mathbb{P}(\|u(t) - v(t)\| \to 0,\ t \to \infty) = 1$
3) $\exists$ "coupling" st $\mathbb{P}(\quad) > 0$