Many body quantum dynamics and nonlinear dispersive PDE

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Main objectives of the lectures

1. To give a short review of the derivation of the NLS from quantum many body systems via the Gross-Pitaevskii (GP) hierarchy.[1]

2. The most involved part in such a derivation of NLS consists in establishing uniqueness of solutions to the GP, which was originally obtained by Erdös-Schlein-Yau. We will focus on approaches to the uniqueness step that are motivated by the perspective coming from nonlinear dispersive PDE, including:
   - the approach of Klainerman-Machedon
   - the approach that we developed with T. Chen, C. Hainzl and R. Seiringer based on the quantum de Finetti's theorem.

[1] The GP hierarchy is an infinite system of coupled linear non-homogeneous PDE.

Outline

Interacting bosons

The mathematical analysis of interacting Bose gases is a hot topic in Math Physics. One of the important research directions is:

- Proof of Bose-Einstein condensation

Bose-Einstein condensation

At very low temperatures dilute Bose gases are characterized by the "macroscopic occupancy of a single one-particle state".

- **The prediction** in 1920's
  Bose, Einstein
- **The first experimental realization** in 1995
  Cornell-Wieman et al, Ketterle et al
- **Proof of Bose-Einstein condensation** around 2000
  Aizenman-Lieb-Seiringer-Solovej-Yngvason, Lieb-Seiringer, Lieb-Seiringer-Yngvason

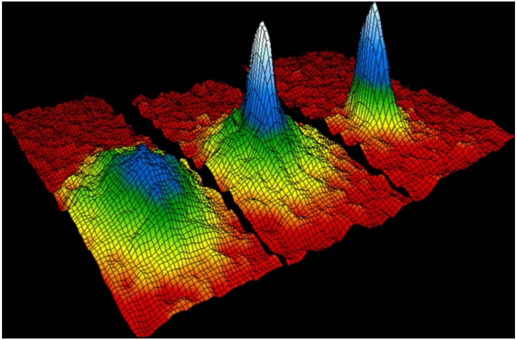

Figure : Velocity distribution data for a gas of rubidium atoms before/just after the appearance of a Bose-Einstein Condensate, and after further evaporation. The photo is a courtesy of Wikipedia.

Nonlinear Schrödinger equation (NLS)

The mathematical analysis of solutions to the nonlinear Schrödinger equation (NLS) has been a hot topic in PDE.

NLS is an example of a **dispersive**[2] equation.

[2] Informally, "dispersion" means that different frequencies of the equation propagate at different velocities, i.e. the solution disperses over time.

The Cauchy problem for a nonlinear Schrödinger equation

$$iu_t + \Delta u = \mu |u|^{p-1} u \tag{1.1}$$

$$u(x,0) = u_0(x) \in H^s(\Omega^n), \ t \in \mathbb{R}, \tag{1.2}$$

where Ω^n is either the space $\mathbb{R}^n$ or the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The equation (1.1) is called

- defocusing if $\mu = 1$
- focusing if $\mu = -1$.

NLS - basic questions - I

1. **Local in time well-posedness, LWP** (existence of solutions, their uniqueness and continuous dependance on initial data[3])
   - How: usually a fixed point argument.
   - Tools: Strichartz estimates
   - Then (in the '80s, '90s):
     - via Harmonic Analysis (e.g. *Kato, Cazenave-Weissler, Kenig-Ponce-Vega*)
     - via Analytic Number Theory (e.g. *Bourgain*)
     - via Probability (e.g. *Bourgain* a.s. LWP[4])
   - Now:
     - via Probability (e.g. *Burq-Tzvetkov, Rey-Bellet - Nadmoh - Oh - Staffilani, Nahmod-Staffilani, Bourgain-Bulut*)
     - via Incidence Theory (a hot new direction *Bourgain-Demeter*)

[3] LWP: For any $u_0 \in X$ there exist $T > 0$ and a unique solution u to the IVP in $C([0,T],X)$ that is also stable in the appropriate topology.

[4] a.s. LWP: There exists $Y \subset X$, with $\mu(Y) = 1$ and such that for any $u_0 \in Y$ there exist $T > 0$ and a unique solution u to the IVP in $C([0,T],X)$ that is also stable in the appropriate topology.

NLS - more on local well-posedness

(A) **Energy methods:** integrate by parts the IVP to obtain an apriori bound $\sup_{0\leq t\leq T}\|u(\cdot,t)\|_{H^s} \leq C(T,u_0)$. Then use approximative methods to obtain a sequence for which the bound is valid and take a weak limit.
Bad news: usually too many derivatives are needed.

(B) **Iterative methods:** by the Duhamel's formula the IVP

$$iu_t + Lu = N(u)$$

is equivalent to the integral equation

$$u(t) = U(t)u_0 + \int_0^t U(t-\tau)N(u(\tau))d\tau,$$

where $U(t)$ is the solution operator associated to the linear problem.
Tools: Strichartz estimates (*Strichartz, Ginibre-Velo, Yajima, Keel-Tao*)
For any admissible pairs (q,r) and $(\tilde{q},\tilde{r})$ we have

(1.3) $$\|U(t)u_0\|_{L^q_t L^r_x} \leq C\|u_0\|_{L^2_x}.$$

(1.4) $$\|\int_0^t U(t-\tau)N(\tau)\,d\tau\|_{L^q_t L^r_x} \leq C\|N\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}.$$

Good news: one can treat problems with much less regularity.
Bad news: some smallness is needed (e.g. short times or small data).

NLS - basic questions - II

2 **Global in time well-posedness/blow-up**

- How: LWP $+$ use of conserved quantities
- Tools: very technical clever constructions in order to access conserved quantities
- Then (in the '00s):
  - via Harmonic Analysis (e.g. *Bourgain* and *Colliander-Keel-Staffilani-Takaoka-Tao* induction on energy, *Kenig-Merle* concentration-compactness, *Killip - Visan*)
- Now:
  - via Probability (a construction of Gibbs measure e.g. *Burq-Tzvetkov, Oh, Rey-Bellet - Nadmoh - Oh - Staffilani, Bourgain-Bulut*).

Bosons and NLS

What is a connection between:

- interacting bosons

  and

- NLS?

Rigorous derivation of the NLS from quantum many body systems

- How: the topic of these lectures
- Then (in the late '70s and the '80s):
  - via Quantum Field Theory (*Hepp, Ginibre-Velo*)
  - via Math Physics (*Spohn*)
- Now:
  - via Quantum Field Theory (*Rodnianski-Schlein, Grillakis-Machedon-Margetis, Grillakis-Machedon, X. Chen*)
  - via Math Physics (*Fröhlich-Tsai-Yau, Bardos-Golse-Mauser, Erdös-Yau, Adami-Bardos-Golse-Teta, Elgart-Erdös-Schlein-Yau, Erdös-Schlein-Yau*)
  - via Math Physics + Dispersive PDE (*Klainerman-Machedon, Kirkpatrick-Schlein-Staffilani, Chen-P., Chen-P.-Tzirakis, Gressman-Sohinger-Staffilani, Sohinger, X. Chen, X. Chen-Holmer, X. Chen-Smith, Chen-Hainzl-P.-Seiringer, Hong-Taliaferro-Xie, Herr-Sohinger, Bulut*)

From bosons to NLS following Erdös-Schlein-Yau [2006-07]

Step 1: From N-body Schrödinger to BBGKY hierarchy

The starting point is **a system of N bosons whose dynamics is generated by the Hamiltonian**

$$H_N := \sum_{j=1}^{N}(-\Delta_{x_j}) + \frac{1}{N}\sum_{1\leq i<j\leq N} V_N(x_i - x_j)\,, \tag{2.1}$$

on the Hilbert space $\mathcal{H}_N = L^2_{sym}(\mathbb{R}^{dN})$, whose elements $\Psi(x_1, \ldots, x_N)$ are fully symmetric with respect to permutations of the arguments x_j.

Here

$$V_N(x) = N^{d\beta} V(N^\beta x),$$

with $0 < \beta \leq 1$.

When $\beta = 1$, the Hamiltonian

$$H_N := \sum_{j=1}^{N}(-\Delta_{x_j}) + \frac{1}{N}\sum_{1\leq i<j\leq N} V_N(x_i - x_j)\,, \tag{2.2}$$

is called the Gross-Pitaevskii Hamiltonian.

- We note that physically (2.2) describes a very dilute gas, where **interactions among particles are very rare and strong**.
- This is in contrast to a mean field Hamiltonian, where each particle usually reacts with all other particles via a very weak potential.
- However thanks to the factor $\frac{1}{N}$ in front of the interaction potential, (2.2) can be formally interpreted as a mean field Hamiltonian. In particular, one can still apply to (2.2) similar mathematical methods as in the case of a mean field potential.

Schrödinger equation

The wave function satisfies the Schrödinger equation

$$i\partial_t \psi_N = H_N \psi_N \,, \tag{2.3}$$

with initial condition $\Psi_{N,0} \in \mathcal{H}_N$.

- Since the Schrödinger equation (2.3) is linear and the Hamiltonian H_N is self-adjoint, global well-posedness of (2.3) is not an issue.

On the N-body Schrödinger equation

Bad news:

- Qualitative and quantitative properties of the solution are hard to extract in physically relevant cases when number of particles N is very large (e.g. it varies from 10^3 for very dilute Bose-Einstein samples, to 10^{30} in stars).

Good news:

- Physicists often care about macroscopic properties of the system, which can be obtained from averaging over a large number of particles.
- Further simplifications are related to obtaining a macroscopic behavior in the limit as $N \to \infty$, with a hope that the limit will approximate properties observed in the experiments for a very large, but finite N.

To study the limit as $N \to \infty$, one introduces:

- **the N-particle density matrix**

$$\gamma_N(t, \underline{x}_N; \underline{x}'_N) = \Psi_N(t, \underline{x}_N)\overline{\Psi_N(t, \underline{x}'_N)},$$

- **and its k-particle marginals**

$$\gamma_N^{(k)}(t, \underline{x}_k; \underline{x}'_k) = \int d\underline{x}_{N-k}\gamma_N(t, \underline{x}_k, \underline{x}_{N-k}; \underline{x}'_k, \underline{x}_{N-k}),$$

for $k = 1, \dots, N$.

Here

$$\underline{x}_k = (x_1, \dots, x_k),$$
$$\underline{x}_{N-k} = (x_{k+1}, \dots, x_N).$$

The BBGKY[5], hierarchy is given by

$$
\begin{aligned}
i\partial_t \gamma_N^{(k)} &= -(\Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})\gamma_N^{(k)} \\
&+ \frac{1}{N} \sum_{1 \le i < j \le k} \left(V_N(x_i - x_j) - V_N(x'_i - x'_j) \right) \gamma_N^{(k)} \qquad (2.4) \\
&+ \frac{N-k}{N} \sum_{i=1}^{k} \mathrm{Tr}_{k+1} \left(V_N(x_i - x_{k+1}) - V_N(x'_i - x_{k+1}) \right) \gamma_N^{(k+1)} \qquad (2.5)
\end{aligned}
$$

In the limit $N \to \infty$, the sums weighted by combinatorial factors have the following size:

- In (2.4), $\frac{k^2}{N} \to 0$ for every fixed k and sufficiently small β.
- In (2.5), $\frac{N-k}{N} \to 1$ for every fixed k and $V_N(x_i - x_j) \to b_0 \delta(x_i - x_j)$, with $b_0 = \int dx\, V(x)$.

[5] Bogoliubov-Born-Green-Kirkwood-Yvon

Step 2: BBGKY hierarchy → GP hierarchy

As $N \to \infty$, one obtains the infinite GP hierarchy as a <u>weak limit</u>.

$$i\partial_t \gamma_\infty^{(k)} = -\sum_{j=1}^{k} (\Delta_{x_j} - \Delta_{x_j'})\, \gamma_\infty^{(k)} + b_0 \sum_{j=1}^{k} B_{j;k+1} \gamma_\infty^{(k+1)}$$

where the **"contraction operator"** is given via

$$\begin{aligned} &\left(B_{j;k+1}\gamma_\infty^{(k+1)}\right)(t, x_1, \dots, x_k; x_1', \dots, x_k') \\ &\quad = \gamma_\infty^{(k+1)}(t, x_1, \dots, x_j, \dots, x_k, x_j; x_1', \dots, x_k', x_j) \\ &\quad - \gamma_\infty^{(k+1)}(t, x_1, \dots, x_k, x_j'; x_1', \dots, x_j', \dots, x_k', x_j'). \end{aligned}$$

Step 3: Factorized solutions of the GP hierarchy

It is easy to see that

$$\gamma_\infty^{(k)} = |\,\phi\,\rangle\langle\,\phi\,|^{\otimes k} := \prod_{j=1}^{k} \phi(t, x_j)\,\overline{\phi(t, x_j')}$$

is a solution of the GP if ϕ satisfies the cubic NLS

$$i\partial_t\phi + \Delta_x\phi - b_0\,|\phi|^2\,\phi = 0$$

with $\phi_0 \in L^2(\mathbb{R}^d)$.

Step 4: Uniqueness of solutions to the GP hierarchy

While the existence of factorized solutions can be easily obtained, the proof of **uniqueness of solutions** of the GP hierarchy is the most difficult[6] part in this analysis.

[6] We will describe those difficulties soon.

Summary of the method of ESY

Roughly speaking, the method of *Erdös, Schlein, and Yau* for deriving the cubic NLS justifies the heuristic explained above and it consists of the following two steps:

(i) **Deriving the GP hierarchy as the limit as $N \to \infty$ of the BBGKY hierarchy**.

(ii) **Proving uniqueness of solutions for the GP hierarchy,** which implies that for factorized initial data, the solutions of the GP hierarchy are determined by a cubic NLS. The proof of uniqueness is accomplished by using highly sophisticated **Feynman graphs**.

A remark about ESY solutions of the GP hierarchy

- Solutions of the GP hierarchy are studied in "L^1-type trace Sobolev" spaces of k-particle marginals

$$\{\gamma^{(k)} \mid \|\gamma^{(k)}\|_{\mathfrak{h}^1} < \infty\}$$

with norms

$$\|\gamma^{(k)}\|_{\mathfrak{h}^\alpha} := \mathrm{Tr}(|S^{(k,\alpha)}\gamma^{(k)}|)\,,$$

where[7]

$$S^{(k,\alpha)} := \prod_{j=1}^{k} \langle \nabla_{x_j} \rangle^{\alpha} \langle \nabla_{x_j'} \rangle^{\alpha}\,.$$

[7] Here we use the standard notation: $\langle y \rangle := \sqrt{1+y^2}$.

Why is it difficult to prove uniqueness?

One considers the r-fold iterate of the Duhamel formula for $\gamma^{(k)}$, with initial data $\gamma_0^{(k)} = 0$, for some arbitrary $r \in \mathbb{N}$,

$$\gamma^{(k)}(t) = (i\lambda)^r \int_{t \geq t_1 \geq \cdots \geq t_r} dt_1 \cdots dt_r U^{(k)}(t - t_1) B_{k+1} U^{(k+1)}(t_1 - t_2) \cdots \cdots U^{(k+r-1)}(t_{r-1} - t_r) B_{k+r} \gamma^{(k+r)}(t_r)$$

$$=: \int_{t \geq t_1 \geq \cdots \geq t_r} dt_1 \cdots dt_r J^k(\underline{t}_r) \quad , \quad \underline{t}_r := (t_1, \ldots, t_r) . \tag{2.6}$$

A key difficulty stems from the fact that the interaction operator $B_{\ell+1}$ is the sum of $O(\ell)$ terms, therefore (2.6) contains $O(\frac{(k+r-1)!}{(k-1)!}) = O(r!)$ terms.

Uniqueness of GP following Klainerman-Machedon

Klainerman and Machedon (2008) introduced an alternative method for proving uniqueness in a space of density matrices equipped with the Hilbert-Schmidt type Sobolev norm

$$\|\gamma^{(k)}\|_{H_k^\alpha} := \|S^{(k,\alpha)}\gamma^{(k)}\|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})}.$$

The method is based on:

- a reformulation of the relevant combinatorics via the **"board game argument"** and
- the use of certain **space-time estimates** of the type:

$$\|B_{j;k+1}\ U^{(k+1)}\gamma^{(k+1)}\|_{L_t^2\dot{H}^\alpha(\mathbb{R}\times\mathbb{R}^{dk}\times\mathbb{R}^{dk})} \lesssim \|\gamma^{(k+1)}\|_{\dot{H}^\alpha(\mathbb{R}^{d(k+1)}\times\mathbb{R}^{d(k+1)})}.$$

The method of *Klainerman and Machedon* makes the assumption that the a priori space-time bound

$$\|B_{j;k+1}\gamma^{(k+1)}\|_{L^1_t \dot{H}^1_k} < C^k, \quad (2.7)$$

holds, with C independent of k.

Subsequently:

- *Kirkpatrick, Schlein and Staffilani* (2011) were the first to use the KM formulation to derive the cubic NLS in $d = 2$ via proving that the limit of the BBGKY satisfies (2.7).
- *Chen-P* (2011) generalized this to derive the quintic GP in $d = 1, 2$.
- *Xie* (2013) generalized it further to derive a NLS with a general power-type nonlinearity in $d = 1, 2$.
- A derivation of the cubic NLS in $d = 3$ based on the KM combinatorial formulation was settled recently (*Chen-P*; *X. Chen*, *X. Chen-Holmer* and *T. Chen-Taliaferro*).

Uniqueness argument of Klainerman and Machedon

Main steps of the approach:

1. a reformulation of the relevant combinatorics of ESY via the **“board game argument”**
2. the use of certain **space-time estimates**

The board game combinatorial argument, in a nutshell

Revisit the iterated Duhamel formula

Let us go back to the r-fold iterate of the Duhamel formula for $\gamma^{(k)}$, with initial data $\gamma_0^{(k)} = 0$, for some arbitrary $r \in \mathbb{N}$,

$$\gamma^{(k)}(t) = (i\lambda)^r \int_{t \geq t_1 \geq \cdots \geq t_r} dt_1 \cdots dt_r U^{(k)}(t - t_1) B_{k+1} U^{(k+1)}(t_1 - t_2) \cdots$$
$$\cdots U^{(k+r-1)}(t_{r-1} - t_r) B_{k+r} \gamma^{(k+r)}(t_r)$$

(2.8) $$=: \int_{t \geq t_1 \geq \cdots \geq t_r} dt_1 \cdots dt_r J^k(\underline{t}_r) \quad , \quad \underline{t}_r := (t_1, \ldots, t_r) \, .$$

Recalling that $B_{\ell+1} = \sum_{j=1}^{\ell} B_{j;\ell+1}$, we write

$$J^k(\underline{t}_r) = \sum_{\sigma \in \mathcal{M}_{k,r}} J^k(\sigma; \underline{t}_r), \tag{2.9}$$

where

$$J^k(\sigma;\underline{t}_r) := (i\lambda)^r U^{(k)}(t-t_1) B_{\sigma(k+1),k+1} U^{(k+1)}(t_1-t_2)\cdots$$
$$\cdots U^{(k+\ell-1)}(t_{\ell-1}-t_\ell) B_{\sigma(k+\ell),k+\ell}\cdots U^{(k+r-1)}(t_{r-1}-t_r) B_{\sigma(k+r),k+r}\gamma^{(k+r)}(t_r),$$

and σ is a map $\sigma : \{k+1, k+2, ..., k+r\} \to \{1, 2, ..., k+r-1\}$, $\sigma(2) = 1$, and $\sigma(j) < j$ for all j.

Each map σ can be represented by highlighting one nonzero entry in each column of an $(k+r-1)\times r$ matrix:

$$\begin{bmatrix}
\mathbf{B_{1,k+1}} & B_{1,k+2} & \cdots & \cdots & \cdots & \mathbf{B_{1,k+r}} \\
\cdots & \mathbf{B_{2,k+2}} & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \mathbf{B_{\sigma(k+\ell),k+\ell}} & \cdots & \cdots \\
B_{k,k+1} & B_{k,k+2} & \cdots & \cdots & \cdots & \cdots \\
0 & B_{k+1,k+2} & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & 0 & \cdots & \cdots \\
0 & 0 & \cdots & 0 & \cdots & B_{k+r-1,k+r}
\end{bmatrix}. \tag{2.10}$$

A representation of $\gamma^{(k)}$

Having defined σ, we can rewrite $\gamma^{(k)}$ as

$$(2.11) \qquad \gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{M}_{k,r}} \int_{t \geq t_1 \geq \cdots \geq t_r} J^k(\sigma, \underline{t}_r) \, dt_1 ... dt_r \, .$$

where the time domains are given by the same simplex $\{t > t_1 > \cdots > t_r\} \subset [0, t]^r$ for all integrals in the sum over σ.

A very brief summary of the combinatorial part of KM:

- introduce a boardgame on a set related to $\mathcal{M}_{k,r}$, so that an **acceptable move** does not change values of corresponding integrals
- in finitely many acceptable moves, each matrix can be transforrmed to an upper echelon matrix
- an upper echelon matrix is a representative of a class of equivalance
- easy to obtain the number of classes of equivalence
- in each equivalence class, one can re-organize all relevant integrals

Now, the details of the combinatorial argument of KM

Enlarged matrix:

Now we consider the integrals with permuted time integration orders:

$$I(\sigma,\pi) = \int_{t \geq t_{\pi(1)} \geq \dots \geq t_{\pi(r)}} J^k(\sigma; \underline{t}_r)\, dt_1 \dots dt_r, \tag{2.12}$$

where π is a permutation of $\{1, 2, ..., r\}$.

One can associate to $I(\sigma,\pi)$ the matrix

$$\begin{bmatrix} t_{\pi^{-1}(1)} & t_{\pi^{-1}(2)} & \cdots & t_{\pi^{-1}(r)} \\ \mathbf{B_{1,k+1}} & B_{1,k+2} & \cdots & \mathbf{B_{1,k+r}} \\ \cdots & \mathbf{B_{2,k+2}} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ B_{k,k+1} & B_{k,k+2} & \cdots & \cdots \\ 0 & B_{k+1,k+2} & \cdots & \cdots \\ \cdots & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{k+r-1,k+r} \end{bmatrix}$$

whose columns are labeled 1 through r and whose rows are labeled $0, 1, ..., k+r-1$.

On the set of such matrices

$$\begin{bmatrix} t_{\pi^{-1}(1)} & t_{\pi^{-1}(2)} & \cdots & t_{\pi^{-1}(r)} \\ B_{1,k+1} & \mathbf{B_{1,k+2}} & \cdots & \mathbf{B_{1,k+r}} \\ \mathbf{B_{2,k+1}} & B_{2,k+2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ B_{k,k+1} & B_{k,k+2} & \cdots & \cdots \\ 0 & B_{k+1,k+2} & \cdots & \cdots \\ \cdots & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{k+r-1,k+r} \end{bmatrix}$$

KM introduce the following *board game*:

An *acceptable move* is characterized via: If $\sigma(k+\ell) < \sigma(k+\ell-1)$, the player is allowed to do the following three changes at the same time:

- exchange the highlights in columns ℓ and $\ell+1$,
- exchange the highlights in rows $k+\ell-1$ and $k+\ell$,
- exchange $t_{\pi^{-1}(\ell)}$ and $t_{\pi^{-1}(\ell+1)}$.

The main property of the integrals $I(\sigma, \pi)$ is *invariance under acceptable moves*:

Lemma

If (σ, π) is transformed into (σ', π') by an acceptable move, then $I(\sigma, \pi) = I(\sigma', \pi')$.

Upper echelon form

We say that a matrix of the type (2.10) is in *upper echelon form if each highlighted entry in a row is to the left of each highlighted entry in a lower row*.

For example, the following matrix is in upper echelon form (with $k = 1$ and $r = 4$):

$$\begin{bmatrix} \mathbf{B_{1,2}} & B_{1,3} & B_{1,4} & B_{1,5} \\ 0 & \mathbf{B_{2,3}} & B_{2,4} & B_{2,5} \\ 0 & 0 & \mathbf{B_{3,4}} & \mathbf{B_{3,5}} \\ 0 & 0 & 0 & B_{4,5} \end{bmatrix}.$$

Why are upper echelon matrices handy?

The following *normal form* property holds:

Lemma

For each matrix in $\mathcal{M}_{k,r}$, there is a finite number of acceptable moves that transforms the matrix into upper echelon form.

And we can count:

Lemma

Let $C_{k,r}$ denote the number of upper echelon matrices of size $(k+r-1)\times r$. Then

$$C_{k,r} \leq 2^{k+r} \,. \tag{2.13}$$

Let $\mathcal{N}_{k,r}$ denote the subset of matrices in $\mathcal{M}_{k,r}$ which are in upper echelon form. Let σ_s account for a matrix in $\mathcal{N}_{k,r}$. We write $\sigma \sim \sigma_s$ if the matrix corresponding to σ can be transformed into that corresponding to σ_s in finitely many acceptable moves.

Then, the following key theorem holds:

Theorem

Suppose $\sigma_s \in \mathcal{N}_{k,r}$. Then, there exists a subset of $[0,t]^r$, denoted by $D(\sigma_s,t)$, such that

$$(2.14)\ \sum_{\sigma \sim \sigma_s} \int_{t \geq t_1 \geq \cdots \geq t_r} J^k(\sigma; \underline{t}_r)\, dt_1 \ldots dt_r = \int_{D(\sigma_s,t)} J^k(\sigma_s; \underline{t}_r)\, dt_1 \ldots dt_r\,.$$

The space-time estimate

Strichartz-type estimate for the GP

Theorem

Let $\gamma^{(k+1)}$ be the solution of

$$i\partial_t \gamma^{(k+1)}(t, \underline{x}_{k+1}; \underline{x}'_{k+1}) + (\Delta_{\underline{x}_{k+1}} - \Delta_{\underline{x}'_{k+1}})\gamma^{(k+1)}(t, \underline{x}_{k+1}; \underline{x}'_{k+1}) = 0$$

with initial condition $\gamma^{(k+1)}(0, \cdot) = \gamma_0^{(k+1)} \in H^1$. Then, there exists a constant C such that

$$\left\| B_{j;k+1}\gamma^{(k+1)} \right\|_{L^2(\mathbb{R})\dot{H}^1_k(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} \leq C \left\| \gamma_0^{(k+1)} \right\|_{\dot{H}^1_{k+1}(\mathbb{R}^{d(k+1)}\times\mathbb{R}^{d(k+1)})}$$

holds.

In other words: $\left\| B_{j;k+1} U^{(k+1)} \gamma_0^{(k+1)} \right\|_{L^2 \dot{H}^1_k} \leq C \left\| \gamma_0^{(k+1)} \right\|_{\dot{H}^1_{k+1}}$.

The finale

For $I = [0, T]$, $D \subset I^r$ and $D_{t_1} = \{(t_2, ..., t_r) \,|\, (t_1, ..., t_r) \in D\}$ we have that $\|\, \gamma^{(k)}(t) \,\|_{\dot H^1}$ is bounded by the sum of at most 2^{k+r} terms of the form:

$$\left\| \int_D dt_1 ... dt_r U^{(k)}(t - t_1) B_{\sigma(k+1),k+1} U^{(k+1)}(t_1 - t_2) ... B_{\sigma(k+r),k+r} \gamma^{(k+r)} \right\|_{\dot H^1_k}$$

$$= \left\| \int_0^t dt_1 \, U^{(k)}(t - t_1) \int_{D_{t_1}} dt_2 \cdots dt_r B_{\sigma(k+1),k+1} U^{(k+1)}(t_1 - t_2) \cdots B_{\sigma(k+r),k+r} \gamma^{(k+r)} \right\|_{\dot H^1_k}$$

$$\leq \int_{I^r} dt_1 \cdots dt_r \left\| B_{\sigma(k+1),k+1} U^{(k+1)}(t_1 - t_2) \cdots B_{\sigma(k+r),k+r} \gamma^{(k+r)} \right\|_{\dot H^1_k}$$

(2.15)

$$\leq t^{1/2} \int_{I^{r-1}} dt_2 \cdots dt_r \left\| B_{\sigma(k+1),k+1} U^{(k+1)}(t_1 - t_2) B_{\sigma(k+2),k+2} \cdots B_{\sigma(k+r),k+r} \gamma^{(k+r)} \right\|_{L^2_{t_1 \in I} \dot H^1_k}$$

(2.16)

$$\leq t^{1/2} \int_{I^{r-1}} dt_2 \cdots dt_r \left\| B_{\sigma(k+2),k+2} U^{(k+2)}(t_2 - t_3) \cdots B_{\sigma(k+r),k+r} \gamma^{(k+r)} \right\|_{\dot H^1_{k+1}}$$

$$\leq \cdots \left(t^{1/2}\right)^{r-1} \int_I dt_r \left\| B_{\sigma(k+r),k+r} \gamma^{(k+r)} \right\|_{\dot H^1_{k+r-1}} .$$

(2.15) was obtained by Cauchy-Schwarz w.r.t. to t_1, (2.16) via the space-time estimate, and the last line via iteration.

Therefore, we have:

$$\| \gamma^{(k)}(t) \|_{\dot{H}^1} \leq c \left(Ct^{1/2} \right)^{r-1} \int_I dt_r \left\| B_{\sigma(k+r),k+r} \gamma^{(k+r)} \right\|_{\dot{H}^1_{k+r-1}} \tag{2.17}$$

which after we recall the assumption of Klainerman and Machedon

$$\| B_{j;k+1} \gamma^{(k+1)} \|_{L^1_t \dot{H}^1_k} < C^k \tag{2.18}$$

implies

$$\| \gamma^{(k)}(t) \|_{\dot{H}^1} \leq c \left(Ct^{1/2} \right)^{r-1} .$$

Hence by choosing $Ct^{1/2} < 1$ and letting $r \to \infty$, it follows that:

$$\| \gamma^{(k)}(t) \|_{\dot{H}^1} = 0.$$

Going backwards i.e. from NLS to bosons

Since the GP

- arises in a derivation of the NLS from quantum many-body system

it is natural to ask:

1. Whether the GP retains some of the features of a dispersive PDE?
2. Whether methods of nonlinear dispersive PDE can be "lifted" to the GP and the QFT levels?

Some questions about the GP - inspired by the NLS theory

1. **Local in time existence** of solutions to GP.
2. **Blow-up** of solutions to the focusing GP hierarchies.
3. **Global existence** of solutions to the GP hierarchy.
4. **Derivation of the cubic GP hierarchy** in [KM] spaces.
5. **Uniqueness of the cubic GP hierarchy** on $\mathbb{T}^3$.
6. **Uniqueness of the cubic GP hierarchy** on $\mathbb{R}^3$ via dispersive tools.

Now, we shall look a bit into two of those questions:

1. Local in time existence and uniqueness for the GP
2. Negative energy blow-up result for the GP

Local in time existence and uniqueness

The work of *Klainerman and Machedon* inspired us to study the Cauchy problem for GP hierarchies.

Towards a well-posedness result for the GP

Problem: The equations for $\gamma^{(k)}$ do not close & no fixed point argument.

Solution: Endow the space of sequences

$$\Gamma := (\gamma^{(k)})_{k\in\mathbb{N}}.$$

with a suitable topology.

Revisiting the GP hierarchy

Recall,

$$\Delta_{\pm}^{(k)} = \Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k}, \quad \text{with} \quad \Delta_{\underline{x}_k} = \sum_{j=1}^{k} \Delta_{x_j}.$$

We introduce the notation:

$$\Gamma = (\, \gamma^{(k)}(\, t, x_1, \ldots, x_k; x'_1, \ldots, x'_k)\,)_{k\in\mathbb{N}} \,,$$

$$\widehat{\Delta}_{\pm}\Gamma := (\, \Delta_{\pm}^{(k)} \gamma^{(k)}\,)_{k\in\mathbb{N}} \,,$$

$$\widehat{B}\Gamma := (\, B_{k+1}\gamma^{(k+1)}\,)_{k\in\mathbb{N}} \,.$$

Then, the cubic GP hierarchy can be written as[8]

(3.1) $$i\partial_t \Gamma + \widehat{\Delta}_{\pm}\Gamma = \mu \widehat{B}\Gamma \,.$$

[8] Moreover, for $\mu = 1$ we refer to the GP hierarchy as defocusing, and for $\mu = -1$ as focusing.

Spaces

Let

$$\mathfrak{G} := \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$$

be the space of sequences of density matrices

$$\Gamma := (\gamma^{(k)})_{k \in \mathbb{N}}.$$

As a crucial ingredient of our arguments, we introduce Banach spaces $\mathcal{H}_\xi^\alpha = \{ \Gamma \in \mathfrak{G} \mid \| \Gamma \|_{\mathcal{H}_\xi^\alpha} < \infty \}$ where

$$\| \Gamma \|_{\mathcal{H}_\xi^\alpha} := \sum_{k \in \mathbb{N}} \xi^k \| \gamma^{(k)} \|_{H^\alpha(\mathbb{R}^{dk} \times \mathbb{R}^{dk})} .$$

Properties:

- **Finiteness:** $\| \Gamma \|_{\mathcal{H}_\xi^\alpha} < C$ implies that $\| \gamma^{(k)} \|_{H^\alpha(\mathbb{R}^{dk} \times \mathbb{R}^{dk})} < C\xi^{-k}$.
- **Interpretation:** ξ^{-1} upper bound on typical H^α-energy per particle.

With T. Chen, we prove **local in time existence and uniqueness** of solutions to the cubic and quintic GP hierarchy with focusing or defocusing interactions, in a subspace of $\mathcal{H}_\xi^\alpha$, for $\alpha \in \mathfrak{A}(d,p)$, which satisfy a spacetime bound

$$\|\widehat{B}\Gamma\|_{L^1_{t\in I}\mathcal{H}_\xi^\alpha} < \infty, \tag{3.2}$$

for some $\xi > 0$.

Flavor of the proof:
Note that the GP hierarchy can be formally written as a system of integral equations

$$(3.3) \qquad \Gamma(t) = e^{it\widehat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds\, e^{i(t-s)\widehat{\Delta}_\pm}\widehat{B}\Gamma(s)$$

$$(3.4) \qquad \widehat{B}\Gamma(t) = \widehat{B}\, e^{it\widehat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds\, \widehat{B}\, e^{i(t-s)\widehat{\Delta}_\pm}\widehat{B}\Gamma(s)\,,$$

where (3.4) is obtained by applying the operator $\widehat{B}$ on the linear non-homogeneous equation (3.3).

We prove the local well-posedness result by applying the fixed point argument in the following space:

$$(3.5) \qquad \mathfrak{W}^\alpha_\xi(I) := \{\, \Gamma \in L^\infty_{t\in I}\mathcal{H}^\alpha_\xi \mid \widehat{B}\Gamma \in L^1_{t\in I}\mathcal{H}^\alpha_\xi \,\},$$

where $I = [0, T]$.

The conserved energy for the GP and an application

It is possible to:

1. **Identify an observable corresponding to the average energy per particle** and prove that it is conserved.
2. Prove, on the L^2 critical and supercritical level, that solutions of focusing GP hierarchies with a negative average energy per particle and finite variance **blow up in finite time**.

Inspired by the spaces of solutions used by ESY, we introduce the spaces

$$\mathfrak{H}_\xi^\alpha = \{ \Gamma \in \mathfrak{G} \,|\, \| \Gamma \|_{\mathfrak{H}_\xi^\alpha} < \infty \}$$

where

$$\| \Gamma \|_{\mathfrak{H}_\xi^\alpha} := \sum_{k \in \mathbb{N}} \xi^k \| \gamma^{(k)} \|_{\mathfrak{h}^\alpha} ,$$

with

$$\|\gamma^{(k)}\|_{\mathfrak{h}^\alpha} \quad := \quad \mathrm{Tr}(\, | S^{(k,\alpha)} \gamma^{(k)} | \,) \, .$$

Conservation of average energy per particle

Theorem (Chen-P-Tzirakis)

Assume $\Gamma(t) \in \mathfrak{H}_\xi^\alpha$, $\alpha \geq 1$, *solves p-GP for* $\mu = \pm 1$.
Define k-particle energy and ξ*-energy of GP,* $0 < \xi < 1$,

$$E_k(\,\Gamma(t)\,) \;:=\; \mathrm{Tr}\Big[\,\sum_{j=1}^{k}(-\frac{1}{2}\Delta_{x_j})\gamma^{(k)}\,\Big] \;+\; \frac{\mu}{p+2}\mathrm{Tr}\Big[\,B^{+}_{k+\frac{p}{2}}\,\gamma^{(k+\frac{p}{2})}\,\Big]$$

$$\mathcal{E}_\xi(\,\Gamma(t)\,) \;:=\; \sum_{k\geq 1}\xi^k\,E_k(\,\Gamma(t)\,)\,.$$

Then, the ξ*-energy is conserved,* $\mathcal{E}_\xi(\,\Gamma(t)\,) = \mathcal{E}_\xi(\,\Gamma(0)\,)$.
In particular, admissibility[a] $\Rightarrow$ *reduction to the one-particle density*

$$\mathcal{E}_\xi(\,\Gamma(t)\,) \;=\; \Big(\sum_{k\geq 1}k\xi^k\,\Big)\,E_1(\,\Gamma(t)\,).$$

[a] We call $\Gamma = (\gamma^{(k)})_{k\in\mathbb{N}}$ admissible if $\gamma^{(k)} = \mathrm{Tr}_{k+1}\gamma^{(k+1)}$ for all $k \in \mathbb{N}$.

Explicit expression for one-particle energy

Let $k_p := 1 + \frac{p}{2}$. Then,

$$E_1(\Gamma) = \mathrm{Tr}[-\frac{1}{2}\Delta_x \gamma^{(1)}] + \frac{\mu}{p+2}\int dx\, \gamma^{(k_p)}(\underbrace{x,\ldots,x}_{k_p};\underbrace{x,\ldots,x}_{k_p})$$

For factorized states $\Gamma(t) = (|\phi(t)\rangle\langle\phi(t)|^{\otimes k})_{k\in\mathbb{N}}$,

$$E_1(\,\Gamma(t)\,) = \frac{1}{2}\|\nabla\phi(t)\|_{L^2}^2 + \frac{\mu}{p+2}\|\phi(t)\|_{L^{p+2}}^{p+2}\,,$$

coincides with energy for NLS

$$i\partial_t\phi + \Delta\phi + \mu|\phi|^p\phi = 0.$$

Blow-up of solutions to the focusing GP hierarchies

Theorem (Chen-P-Tzirakis)

Let $p \geq p_{L^2} = \frac{4}{d}$. Assume that $\Gamma(t) = (\,\gamma^{(k)}(t)\,)_{k\in\mathbb{N}}$ solves the focusing p-GP with $\Gamma(0) \in \mathfrak{H}^1_\xi$ for some $0 < \xi < 1$, and $\mathrm{Tr}(\,x^2\gamma^{(1)}(0)\,) < \infty$.

If $E_1(\,\Gamma(0)\,) < 0$, then the solution $\Gamma(t)$ blows up in finite time.

Skip details...

Zakharov-Glassey's argument for the L^2-critical or supercritical focusing NLS

Consider a solution of

$$i\partial_t \phi = -\Delta\phi - |\phi|^p \phi$$

with $\phi(0) = \phi_0 \in H^1(\mathbb{R}^d)$ and $p \geq p_{L^2} = \frac{4}{d}$, such that

$$E[\phi(t)] := \frac{1}{2}\|\nabla\phi(t)\|_{L^2}^2 - \frac{1}{p+2}\|\phi(t)\|_{L^{p+2}}^{p+2} = E[\phi_0] < 0.$$

Moreover, assume that $\| \, |x|\phi_0 \, \|_{L^2} < \infty$.
Then the quantity $V(t) := \langle \phi(t), x^2\phi(t)\rangle$ satisfies the **virial identity**

$$\partial_t^2 V(t) \; = \; 16E[\phi_0] \; - \; 4d\,\frac{p - p_{L^2}}{p+2}\|\phi(t)\|_{L^{p+2}}^{p+2} \, . \tag{3.6}$$

Hence, if $E[\phi_0] < 0$, and $p \geq p_{L^2}$, this identity shows that V is a strictly concave function of t. But since V is also non-negative, we conclude that the solution can exit only for a finite amount of time.

Zakharov-Glassey's argument for the L^2-critical or supercritical focusing GP

The quantity that will be relevant in reproducing Zakharov-Glassey's argument is given by

$$(3.7) \qquad V_k(\,\Gamma(t)\,) \quad := \quad \mathrm{Tr}(\sum_{j=1}^{k} x_j^2 \gamma^{(k)}(t)\,)\,.$$

Similarly as in our discussion of the conserved energy, we observe that[9]

$$(3.8) \qquad V_k(\,\Gamma(t)\,) = k\;\, V_1(\,\Gamma(t)\,)\,.$$

[9]Again, this follows from the fact that $\gamma^{(k)}$ is symmetric in its variables, and from the admissibility of $\gamma^{(k)}(t)$ for all $k \in \mathbb{N}$.

We calculate $\partial_t^2 V_1(t)$ and relate it to the conserved energy per particle:

$$\partial_t^2 V_1(t) = 16 E_1(\,\Gamma(0)\,) + 4 d\mu \, \frac{p - p_{L^2}}{p+2} \int dX \, \gamma(\underbrace{X, ..., X}_{1+\frac{p}{2}}; \underbrace{X, ..., X}_{1+\frac{p}{2}}).$$

Hence for the focusing ($\mu = -1$) GP hierarchy with $p \geq p_{L^2}$,

$$\partial_t^2 V_1(t) \leq 16 E_1(\,\Gamma(0)\,).$$

However, the function $V_1(t)$ is nonnegative, so we conclude that if $E_1(\,\Gamma(0)\,) < 0$, the solution can exist only for a finite amount of time.

Dispersive tools at the level of the GP

1. Tools at the level of the GP, that are inspired by the NLS techniques, are instrumental in understanding:
   - Well-posedness for the GP hierarchy
   - Well-posedness for quantum many body systems
   - Going from bosons to NLS in Klainerman-Machedon spaces

   Results of: *Gressman-Sohinger-Staffilani, Sohinger, Chen-P, Chen-P-Tzirakis, Chen-Taliaferro, X. Chen, X. Chen-Holmer*.

2. But there were still few questions that resisted the efforts to apply newly built tools at the level of the GP, e.g.
   - Long time behavior of the GP hierarchy
   - Uniqueness of the cubic GP on $\mathbb{T}^3$
   - Uniqueness of the quintic GP on $\mathbb{R}^3$

Q & A session

Q How to address the questions that are “GP tools resistant”?

an **A** Use tools at the level of the NLS?

Q How to use NLS tools when considering the GP?

an **A** Apply **the quantum de Finetti theorem**, which roughly says that (relevant) solutions to the GP are given via an average of factorized solutions.

Quantum de Finetti as a bridge between the NLS and the GP

What is quantum De Finetti?

Strong quantum de Finetti theorem

Due to: *Hudson-Moody (1976/77), Stormer (1969), Lewin-Nam-Rougerie (2013)*

Theorem

(Strong Quantum de Finetti theorem) Let $\mathcal{H}$ be any separable Hilbert space and let $\mathcal{H}^k = \bigotimes_{sym}^k \mathcal{H}$ denote the corresponding bosonic k-particle space. Let Γ denote a collection of admissible bosonic density matrices on $\mathcal{H}$, i.e.,

$$\Gamma = (\gamma^{(1)}, \gamma^{(2)}, \dots)$$

with $\gamma^{(k)}$ a non-negative trace class operator on $\mathcal{H}^k$, and $\gamma^{(k)} = \mathrm{Tr}_{k+1}\gamma^{(k+1)}$, where Tr_{k+1} denotes the partial trace over the $(k+1)$-th factor. Then, there exists a unique Borel probability measure μ, supported on the unit sphere $S \subset \mathcal{H}$, and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus one, such that

$$\gamma^{(k)} = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad , \quad \forall k \in \mathbb{N}\,. \tag{4.1}$$

Weak quantum de Finetti theorem

The limiting hierarchies obtained via weak-* limits from the BBGKY hierarchy of bosonic N-body Schrödinger systems as in *Erdös-Schlein-Yau* do not necessarily satisfy admissibility.

- A weak version of the quantum de Finetti theorem then still applies (a version was recently proven by y *Lewin-Nam-Rougerie*).

De Finetti theorems in action

1. Uniqueness of solutions to the GP hierarchy
2. Scattering for the GP hierarchy

Uniqueness of solutions to the GP via quantum de Finetti theorems

- Until recently, the only available proof of unconditional uniqueness of solutions in[10] $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ to the cubic GP hierarchy in $\mathbb{R}^3$ was given in the works of Erdös, Schlein, and Yau, who developed an approach based on use of Feynman graphs. A key ingredient in their proof is a powerful combinatorial method that resolves the problem of the factorial growth of number of terms in iterated Duhamel expansions.

- Recently, together with T. Chen, C. Hainzl and R. Seiringer, we obtained a new proof based on quantum de Finetti theorem.

[10] The $\mathfrak{H}^1$ denotes the trace class Sobolev space defined for the entire sequence $(\gamma^{(k)})_{k\in\mathbb{N}}$:

$$\mathfrak{H}^1 := \left\{ (\gamma^{(k)})_{k\in\mathbb{N}} \,\middle|\, \mathrm{Tr}(|S^{(k,1)}\gamma^{(k)}|) < M^{2k} \text{ for some constant } M < \infty \right\}.$$

Mild solution to the GP hierarchy

A **mild solution** in the space $L^\infty_{t\in[0,T)}\mathfrak{H}^1$, to the GP hierarchy with initial data $(\gamma^{(k)}(0))_{k\in\mathbb{N}} \in \mathfrak{H}^1$, is a solution of the integral equation

$$\gamma^{(k)}(t) = U^{(k)}(t)\gamma^{(k)}(0) + i\lambda \int_0^t U^{(k)}(t-s)B_{k+1}\gamma^{(k+1)}(s)ds \quad , \quad k \in \mathbb{N},$$

satisfying

$$\sup_{t\in[0,T)} \mathrm{Tr}(|S^{(k,1)}\gamma^{(k)}(t)|) < M^{2k}$$

for a finite constant M independent of k.

Here,

$$U^{(k)}(t) := \prod_{\ell=1}^{k} e^{it(\Delta_{x_\ell} - \Delta_{x'_\ell})}$$

denotes the free k-particle propagator.

Statement of the result

Theorem (Chen-Hainzl-P-Seiringer)

Let $(\gamma^{(k)}(t))_{k\in\mathbb{N}}$ be a mild solution in $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ to the (de)focusing cubic GP hierarchy in $\mathbb{R}^3$ with initial data $(\gamma^{(k)}(0))_{k\in\mathbb{N}} \in \mathfrak{H}^1$, which is either admissible, or obtained at each t from a weak- limit.*
Then, $(\gamma^{(k)})_{k\in\mathbb{N}}$ is the unique solution for the given initial data.
Moreover, assume that the initial data $(\gamma^{(k)}(0))_{k\in\mathbb{N}} \in \mathfrak{H}^1$ satisfy

$$\gamma^{(k)}(0) = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad , \quad \forall k \in \mathbb{N}, \tag{4.2}$$

where μ is a Borel probability measure supported either on the unit sphere or on the unit ball in $L^2(\mathbb{R}^3)$, and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus one. Then,

$$\gamma^{(k)}(t) = \int d\mu(\phi)(|S_t(\phi)\rangle\langle S_t(\phi)|)^{\otimes k} \quad , \quad \forall k \in \mathbb{N}, \tag{4.3}$$

where $S_t : \phi \mapsto \phi_t$ is the flow map of the cubic (de)focusing NLS.

Key tools that we use:

1. **The boardgame combinatorial organization** as presented by *Klainerman and Machedon* (KM)
2. **The quantum de Finetti theorem** allows one to avoid using the condition that was assumed in the work of KM.

Setup of the proof

Assume that we have two positive semidefinite solutions $(\gamma_j^{(k)}(t))_{k\in\mathbb{N}} \in L^\infty_{t\in[0,T)}\mathfrak{H}^1$ satisfying the same initial data,

$$(\gamma_1^{(k)}(0))_{k\in\mathbb{N}} = (\gamma_2^{(k)}(0))_{k\in\mathbb{N}} \in \mathfrak{H}^1.$$

Then,

$$\gamma^{(k)}(t) := \gamma_1^{(k)}(t) - \gamma_2^{(k)}(t) \quad , \quad k \in \mathbb{N}, \tag{4.4}$$

is a solution to the GP hierarchy with initial data $\gamma^{(k)}(0) = 0 \ \forall k \in \mathbb{N}$, and it suffices to prove that

$$\gamma^{(k)}(t) = 0$$

for all $k \in \mathbb{N}$, and for all $t \in [0, T)$.

Remarks:

- From de Finetti theorems, we have

$$\gamma_j^{(k)}(t) = \int d\mu_t^{(j)}(\phi)\big(|\phi\rangle\langle\phi|\big)^{\otimes k} \quad , \quad j = 1, 2 \, ,$$

$$\gamma^{(k)}(t) = \int d\widetilde{\mu}_t(\phi)\big(|\phi\rangle\langle\phi|\big)^{\otimes k} \, , \tag{4.5}$$

  where $\widetilde{\mu}_t := \mu_t^{(1)} - \mu_t^{(2)}$ is the difference of two probability measures on the unit ball in $L^2(\mathbb{R}^3)$.

- From the assumptions of Theorem 9, we have that

$$\sup_{t\in[0,T)} \mathrm{Tr}(|S^{(k,1)}\gamma_i^{(k)}(t)|) < M^{2k} \quad , \quad k \in \mathbb{N} \ , \ i = 1, 2, \tag{4.6}$$

  for some finite constant M, which is equivalent to

$$\int d\mu_t^{(j)}(\phi)\|\phi\|_{H^1}^{2k} < M^{2k} \quad , \quad j = 1, 2 \, , \tag{4.7}$$

  for all $k \in \mathbb{N}$.

Representation of solution using KM and de Finetti

KM implies that we can represent $\gamma^{(k)}(t)$ in upper echelon form:

$$\begin{aligned}\gamma^{(k)}(t) &= \sum_{\sigma\in\mathcal{N}_{k,r}} \int_{D(\sigma,t)} dt_1 \cdots dt_r U^{(k)}(t-t_1)B_{\sigma(k+1),k+1}U^{(k+1)}(t_1-t_2)\cdots \\ &\qquad \cdots U^{(k+r-1)}(t_{r-1}-t_r)B_{\sigma(k+r),k+r}\gamma^{(k+r)}(t_r)\end{aligned}$$

Now using the quantum de Finetti theorem, we obtain:

$$\gamma^{(k)}(t) = \sum_{\sigma\in\mathcal{N}_{k,r}} \int_{D(\sigma,t)} dt_1,\ldots,dt_r \int d\widetilde{\mu}_{t_r}(\phi)\, J^k(\sigma;t,t_1,\ldots,t_r)\,,$$

where

$$\begin{aligned}J^k(\sigma;t,t_1,\ldots,t_r;\underline{x}_k;\underline{x}'_k) = \Big(& U^{(k)}(t-t_1)B_{\sigma(k+1),k+1}U^{(k+1)}(t_1-t_2)\cdots \\ & \cdots U^{(k+r-1)}(t_{r-1}-t_r)B_{\sigma(k+r),k+r}\big(|\phi\rangle\langle\phi|\big)^{\otimes(k+r)}\Big)(\underline{x}_k;\underline{x}'_k)\,.\end{aligned}$$

Product form

For *fixed* ϕ, we note that since

$$(4.8)\qquad \big(|\phi\rangle\langle\phi|\big)^{\otimes(k+r)}(\underline{x}_{k+r};\underline{x}'_{k+r}) = \prod_{i=1}^{k+r}(|\phi\rangle\langle\phi|)(x_i;x_i')$$

is given by a product of 1-particle kernels, it follows that

$$(4.9)\qquad J^k(\sigma;t,t_1,\dots,t_r;\underline{x}_k;\underline{x}'_k) = \prod_{j=1}^{k} J_j^1(\sigma_j;t,t_{\ell_{j,1}},\dots,t_{\ell_{j,m_j}};x_j;x_j')$$

likewise has product form, for each fixed σ.

Goal:

Hence[11]

$$\mathrm{Tr}(\,|\gamma^{(k)}|\,)$$

$$\leq \quad C^r \sum_{i=1,2} \sup_{\sigma} \int_{[0,t]^r} dt_1 \cdots dt_r \int d\mu_{t_r}^{(i)}(\phi) \prod_{j=1}^{k} \mathrm{Tr}\Big(\Big| J_j^1(\sigma_j; t, t_{\ell_{j,1}}, \ldots, t_{\ell_{j,m_j}}) \Big| \Big) .$$

Goal: prove that the right hand side tends to zero as $r \to \infty$, for $t \in [0, T)$, and sufficiently small $T > 0$. Since r is arbitrary, this implies that the left hand side equals zero, thus establishing uniqueness.

[11] Recall that for a fixed k, the number of inequivalent echelon forms is bounded by C^r.

Binary tree graphs

We now **introduce binary tree graphs** as a bookkeeping device to keep track of the complicated contraction structures imposed by the interaction operators inside the iterated Duhamel formula.

Definition of binary trees

We associate to the iterated Duhamel formula the union of k disjoint binary tree graphs, $(\tau_j)_{j=1}^{k}$. We assign:

- An **internal vertex** v_ℓ, $\ell = 1, \dots, r$, to each operator $B_{\sigma(k+\ell),k+\ell}$.
- A **root vertex** w_j, $j = 1, \dots, k$ to each factor $J_j^1(\cdots; x_j; x_j')$ in (4.9).
- A **leaf vertex** u_i, $i = 1, \dots, k+r$, to the factor $(|\phi\rangle\langle\phi|)(x_i; x_i')$ in (4.8).

We say that the tree τ_j is **distinguished** if $v_r \in \tau_j$, and **regular** if $v_r \notin \tau_j$.

How to draw a tree?

For the sake of concreteness, we draw graphs as follows. We consider the strip in $(x, y) \in \mathbb{R}^2$ given by $x \in [0, 1]$ and draw:

- all root vertices $(w_j)_{j=1}^{k}$, ordered vertically, on the line $x = 0$,
- all internal vertices $(v_\ell)_{\ell=1}^{r}$ in the region $x \in (0, 1)$, where $v_{\ell'}$ is on the right of v_ℓ if $\ell' > \ell$.
- all leaf vertices $(u_i)_{i=1}^{k+r}$, ordered vertically, on the line $x = 1$.
- We introduce the equivalence relation "$\sim$" of *connectivity* between vertices to describe the contraction structure determined by $B_{\sigma(k+\ell),k+\ell}$ operators. Between any pair of connected vertices, we draw a connecting line, which we refer to as an *edge*.

A drawing of tree graphs

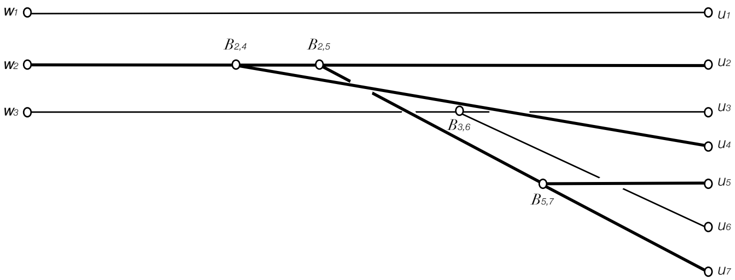

Figure 1. The disjoint union of three tree graphs τ_j, $j = 1, 2, 3$, corresponding to the case $k = 3$, $r = 4$, and

$$J^3(\sigma; t, t_1, \ldots, t_4) = U^{(3)}_{0,1} B_{2,4} U^{(4)}_{1,2} B_{2,5} U^{(5)}_{2,3} B_{3,6} U^{(6)}_{3,4} B_{5,7} (|\phi\rangle\langle\phi|)^{\otimes 7} ,$$

The root vertex w_j belongs to the tree τ_j, $j = 1, 2, 3$. The internal vertices correspond to $v_1 \sim B_{2,4}$, $v_2 \sim B_{2,5}$, $v_3 \sim B_{3,6}$, and $v_4 \sim B_{5,7}$. The leaf vertices u_5 and u_7, and the internal vertex $v_4 \sim B_{5,7}$ are distinguished. The distinguished tree τ_2 is drawn with thick edges.

Roadmap of the proof

1. recognize that a certain product structure gets preserved from right to left (via recursively introducing kernels that account for contractions performed by B operators)

2. get an estimate on integrals in upper echelon form via recursively performing Strichartz estimates (at the level of the Schrödinger equation) from left to right

A taste of the proof or skip tasting

We consider, as an example, the contribution to the main bound of the form

$$\int_{[0,T)^3} dt_1 dt_2 dt_3 \int d\mu_{t_3}^{(i)}(\phi)$$

(4.10)

$$\mathrm{Tr}\Big(\Big| \Big(U^{(1)}(t-t_1) B_{1,2} U^{(2)}(t_1-t_2) B_{2,3} U^{(3)}(t_2-t_3) B_{3,4} \big(|\phi\rangle\langle\phi|\big)^{\otimes 4} \Big| \Big) .$$

Recursive determination of contraction structure

To account for the contractions performed by $B_{\sigma(\alpha+1),\alpha+1}$, we introduce kernels Θ_α, $\alpha = 1, \dots, 3$:

$$\Theta_\alpha(x, x') = \sum_{\beta_\alpha} c^\alpha_{\beta_\alpha} \chi^\alpha_{\beta_\alpha}(x) \overline{\psi^\alpha_{\beta_\alpha}}(x')$$

where $\chi^\alpha_{\beta_\alpha}$, $\psi^\alpha_{\beta_\alpha}$ are certain functions that will be recursively determined, and $c^\alpha_{\beta_\alpha}$ are coefficients with values in $\{1, -1\}$.

The kernel Θ_3

We start at the last interaction operator $B_{3,4}$ in (4.10). It acts nontrivially only on the 3-rd and 4-th factor in $(|\phi\rangle\langle\phi|)^{\otimes 4}$,

(4.11) $$B_{3,4}(|\phi\rangle\langle\phi|)^{\otimes 4} = (|\phi\rangle\langle\phi|)^{\otimes 2} \otimes \Theta_3\,.$$

The kernel Θ_3 is obtained from contracting a two particle density matrix to the one particle density matrix via $B_{1,2}$ (which acts on a two-particle kernel $f(x,y;x',y')$ by $(B_{1,2}f)(x,x') = f(x,x;x',x) - f(x,x';x',x')$),

$$\Theta_3(x,x') := B_{1,2}\Big((|\phi\rangle\langle\phi|)^{\otimes 2}\Big)(x,x') = \widetilde{\psi}(x)\overline{\phi(x')} - \phi(x)\overline{\widetilde{\psi}(x')}$$

(4.12) $$=: \sum_{\beta_3=1}^{2} c^3_{\beta_3}\chi^3_{\beta_3}(x)\overline{\psi^3_{\beta_3}}(x')$$

where

(4.13) $$\widetilde{\psi} := |\phi|^2\phi\,.$$

Here, we have $c^3_1 = 1$, $c^3_2 = -1$, $\chi^3_1 = \widetilde{\psi}$, $\chi^3_2 = \phi$, $\psi^3_1 = \phi$, $\psi^3_2 = \widetilde{\psi}$.

In a similar way, one determines Θ_2 and Θ_1.

Main difficulty

The main difficulty stems from the fact that the term $\widetilde{\psi} = |\phi|^2\phi$ can only be controlled in L^2, where by Sobolev embedding,

$$\|\widetilde{\psi}\|_{L^2} \leq C\|\phi\|_{H^1}^3,$$

which can be controlled by the assumptions of the theorem.

Our objective is to apply the triangle inequality to the trace norm inside (4.10), and to recursively "propagate" the resulting L^2 norm through all intermediate terms until we reach $\widetilde{\psi}$.

Return to (4.10) and perform recursive bounds

• *Integral in t_1.* Applying Cauchy-Schwarz with respect to the integral in t_1 and the triangle inequality for the trace norm, we obtain that

$$\begin{aligned}(4.10) \quad &= \int_{[0,T)^3} dt_1 dt_2 dt_3 \int d\mu_{t_3}^{(i)}(\phi) \mathrm{Tr}\Big(\Big| U^{(1)}(t-t_1)\Theta_1 \Big| \Big) \\ &\leq \sum_{\beta_1=1}^{8} T^{1/2} \int_{[0,T)^2} dt_2 dt_3 \int d\mu_{t_3}^{(i)}(\phi) \Big\| \, \|\chi^1_{\beta_1}\|_{L^2_x} \|\psi^1_{\beta_1}\|_{L^2_x} \Big\|_{L^2_{t_1\in[0,T)}} ,\end{aligned}$$

It can be seen that given $\beta_1 \in \{1, \ldots, 8\}$, there exists β_2 such that

$$\begin{aligned}\chi^1_{\beta_1}(x) &= (U_{1,3}\phi)(x) \\ \psi^1_{\beta_1}(x) &= (U_{1,3}\phi)(x)\overline{(U_{1,2}\chi^2_{\beta_2})}(x)(U_{1,2}\psi^2_{\beta_2})(x)\end{aligned}$$

(or with a cubic expressions for $\chi^1_{\beta_1}$ and a linear expression for $\psi^1_{\beta_1}$). Therefore,

$$\Big\| \, \|\chi^1_{\beta_1}\|_{L^2_x} \|\psi^1_{\beta_1}\|_{L^2_x} \Big\|_{L^2_{t_1\in[0,T)}} = \|\phi\|_{L^2_x} \Big\| (U_{1,3}\phi)(x)\overline{(U_{1,2}\chi^2_{\beta_2})}(x)(U_{1,2}\psi^2_{\beta_2})(x) \Big\|_{L^2_{t_1\in[0,T)} L^2_x} .$$

The crucial estimate (via Strichartz)

Next we observe that

$$\begin{aligned} & \left\| (e^{it\Delta} f_1)(x)\overline{(e^{it\Delta} f_2)(x)}(e^{it\Delta} f_3)(x) \right\|_{L^2_t(\mathbb{R})L^2_x(\mathbb{R}^3)} \\ & \leq \quad \| e^{it\Delta} f_1 \|_{L^\infty_t L^6_x} \| e^{it\Delta} f_2 \|_{L^\infty_t L^6_x} \| e^{it\Delta} f_3 \|_{L^2_t L^6_x} \\ (4.14) \qquad & \leq \quad C \| f_1 \|_{H^1_x} \| f_2 \|_{H^1_x} \| f_3 \|_{L^2_x} \end{aligned}$$

using the Hölder inequality, the Sobolev inequality, and the Strichartz estimate $\| e^{it\Delta} f \|_{L^2_t L^6_x} \leq C \| f \|_{L^2}$ for the free Schrödinger evolution.

We make the important observation that in (4.14), we can place the L^2_x-norm on any of the three functions f_j, $j = 1, 2, 3$, and not only on f_3.

The crucial estimate continues

Similarly, if a derivative is included,

$$\Big\| \nabla_x \big((e^{it\Delta} f_1)(x)\overline{(e^{it\Delta} f_2)}(x)(e^{it\Delta} f_3)(x) \big) \Big\|_{L^2_t(\mathbb{R}) L^2_x(\mathbb{R}^3)}$$

$$\leq \sum_{j=1}^{3} \| e^{it\Delta} \nabla_x f_j \|_{L^2_t L^6_x} \prod_{\substack{1 \leq i \leq 3 \\ i \neq j}} \| e^{it\Delta} f_i \|_{L^\infty_t L^6_x}$$

$$\leq C \, \| f_1 \|_{H^1_x} \| f_2 \|_{H^1_x} \| f_3 \|_{H^1_x} \, , \qquad (4.15)$$

which, together with (4.14), implies that

$$\Big\| (e^{it\Delta} f_1)(x)\overline{(e^{it\Delta} f_2)}(x)(e^{it\Delta} f_3)(x) \Big\|_{L^2_t(\mathbb{R}) H^1_x(\mathbb{R}^3)} \leq C \prod_{j=1}^{3} \| f_j \|_{H^1_x} \, . \qquad (4.16)$$

The integral in t_1 continues

Only one of the factors $\chi^2_{\beta_2}$, $\psi^2_{\beta_2}$ is distinguished[12], say for instance $\psi^2_{\beta_2}$. We then use (4.14) in such a way that the L^2_x-norm is applied to this term, thus obtaning:

$$(4.10) \leq CT^{1/2} \sum_{\beta_1=1}^{8} \int_{[0,T)^2} dt_2 dt_3 \int d\mu^{(i)}_{t_3}(\phi) \|\phi\|^2_{H^1_x} \|\chi^2_{\beta_2}\|_{H^1_x} \|\psi^2_{\beta_2}\|_{L^2_x} ,$$

where the indices β_2 depend on β_1.

Next, we use the defining relation for the functions $\chi^2_{\beta_2}$, $\psi^2_{\beta_2}$, and consider the integral in t_2 and then, at the end, the integral in t_3.

[12] We call a factor *distinguished* if it is a function of $\widetilde{\psi}$.

Using de Finetti for the last step

Subsequently we obtain

$$\begin{aligned} (4.10) \quad &\leq \quad CT \sum_{\beta_1=1}^{8} \int_{[0,T)} dt_3 \int d\mu_{t_3}^{(i)}(\phi) \|\phi\|_{H^1}^5 \|\widetilde{\psi}\|_{L^2_x} \\ &\leq \quad 8CT^2 \sup_{t_3 \in [0,T)} \int d\mu_{t_3}^{(i)}(\phi) \|\phi\|_{H^1}^8 \\ (4.17) \quad &\leq \quad 8CT^2 M^4 , \end{aligned}$$

where we used $\|\widetilde{\psi}\|_{L^2_x} \leq C\|\phi\|_{H^1}^3$ from Sobolev embedding, and the bound related to the de Finetti theorem, which is uniform in t_3.

Scattering for the GP hierarchy

- Previous works: Scattering for the GP has been a longstanding open problem despite much activity in the field.
- Our result, joint with Chen-Hainzl-Seiringer: Establishes the existence of scattering states for the cubic defocusing GP hierarchy on $\mathbb{R}^3$.
- How: Via the de Finetti theorem, the result follows from the scattering theory for the NLS.

Scattering for the NLS

Let us recall that in the defocusing case $\lambda = 1$, the cubic NLS

$$(4.18) \qquad i\partial_t \phi(t) = -\Delta\phi(t) + \lambda|\phi(t)|^2\phi(t) \quad , \quad \phi(0) = \phi_0 \in H^1 \, ,$$

is globally well-posed and displays the existence of scattering states and asymptotic completeness:

Theorem

Let $S_t : \phi_0 \mapsto \phi(t)$ denote the flow map associated to (4.18), for $t \in \mathbb{R}$ and $\lambda = 1$. Then, there exist continuous bijections (wave operators) $W_+, W_- : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$, such that the strong limit

$$(4.19) \qquad \lim_{t\to\pm\infty} e^{-it\Delta} S_t(\phi_0) = \phi_\pm \quad , \quad \phi_0 = W_\pm(\phi_\pm)$$

holds for all $\phi_0 \in H^1(\mathbb{R}^3)$.

The statement of the scattering result

Theorem (Chen-Hainzl-P-Seiringer)

Let $\gamma_0^{(k)} = \int d\mu(\phi)\big(|\phi\rangle\langle\phi|\big)^{\otimes k}$ *and such that*

$$(4.20) \qquad \int d\mu(\phi)(E[\phi])^{2k} \leq R^k, \text{ for some } R > 0, \text{ and all } k \in \mathbb{N}.$$

Let $\gamma^{(k)}(t) = \int d\mu(\phi)(|S_t\phi\rangle\langle S_t\phi|)^{\otimes k}$, *for* $k \in \mathbb{N}$, *denote the unique solution to the cubic defocusing GP satisfying* $\gamma^{(k)}(0) = \gamma_0^{(k)}$, *for* $k \in \mathbb{N}$.
Then, there exist unique asymptotic measures μ_+, μ_- *such that* $\gamma_\pm^{(k)} := \int d\mu_\pm(\phi)(|\phi\rangle\langle\phi|)^k$ *are scattering states on* $L^2(\mathbb{R}^{3k})$ *satisfying*

$$\lim_{t\to\pm\infty} \mathrm{Tr}\Big(\Big| S^{(k,1)}\Big[U^{(k)}(-t)\gamma^{(k)}(t) - \gamma_\pm^{(k)}\Big]\Big|\Big) = 0, \text{ for all } k \in \mathbb{N}.$$

In particular, $d\mu_\pm(\phi) = d\mu(W_\pm(\phi))$ *where the continuous bijections* W_+, $W_- : H^1 \to H^1$ *are the wave operators from Theorem 10.*

On the initial data

- We note that while the de Finetti theorems provide the existence and uniqueness of a measure μ, μ is in general not explicitly known. Therefore, it is important to express the condition (4.20), directly at the level of density matrices.
- This can be done using *higher order energy functionals* for GP hierarchies that were introduced in an earlier work of Chen-P.

The roadmap of the proof

1. Initial conditions are chosen so that μ-almost surely, there exists a unique solution to the defocusing cubic NLS (4.18) with initial data ϕ which exhibits scattering and asymptotic completeness:

   (4.21) $$\lim_{t\to\pm\infty} \|e^{-it\Delta} S_t(\phi) - \phi_\pm\|_{H^1} = 0\,.$$

   Then, $\phi_\pm = W_\pm^{-1}(\phi)$.

2. Define scattering states for the GP as:

   (4.22) $$\gamma_\pm^{(k)} := \int d\mu(\phi)\big(|\phi_\pm\rangle\langle\phi_\pm|\big)^{\otimes k} = \int d\mu_\pm(\phi)\big(|\phi\rangle\langle\phi|\big)^{\otimes k}\,,$$

   where $d\mu_\pm(\phi) = d\mu(W_\pm(\phi))$.

3. Prove the existence of scattering states at the level of the GP using:
   - Our uniqueness theorem for the GP
   - The definition of the scattering states (4.22)
   - Scattering for the NLS (4.21)

Very recent related works

- **Uniqueness of solutions to the cubic GP in low regularity spaces**
  Hong-Taliaferro-Xie
- **Uniqueness of solutions to the quintic GP on $\mathbb{R}^3$**
  Hong-Taliaferro-Xie
- **Uniqueness of solutions to the cubic GP on $\mathbb{T}^d$**
  Sohinger, *Herr-Sohinger*.
- **Uniqueness of solutions to the infinite hierarchy that appears in a connection to the Chern-Simons-Schrödinger system**
  X. Chen-Smith
- **Negative energy blow-up for the focusing Hartree hierarchy**
  Bulut

Back and forth from many body systems to nonlinear equations

Other examples:

- "From Newton to Boltzmann: hard spheres and short-range potentials"
  Gallagher - Saint-Raymond -Texier, 2012
- "Kac's Program in Kinetic Theory"
  Mischler - Mouhot, 2011

Thank you!