Kenji Nakanishi Lecture 2

Today we discuss the proof.

Outline of proof for $V=0$.

(NLS) $i\dot{u} - \Delta u = |u|^2 u$

$u(t,|x|) : \mathbb{R}^{1+3} \to \mathbb{C}$, $u(0) \in H^1_r(\mathbb{R}^3)$

ground state $-\Delta Q + Q = Q^3$

$0 < Q \in H^1_r$, $\Phi \to \{0\}$, $\Psi \to \{e^{i\theta} Q_\omega\}_{\substack{\theta>0 \\ \omega>0}}$

$Q_\omega = \omega^{1/2} Q(\omega^{1/2} x)$

$$u(0) \in \mathcal{H}(\varepsilon) = \{\Phi \in H^1_r \mid M(\Phi)E(\Phi) < M(Q)E(Q) + \varepsilon^2\}$$
$0 < \varepsilon \ll 1$

Scaling: u: sol $\iff$ $\lambda u(\lambda^2 t, \lambda x)$ soln

$M \to \lambda^{-1} M(u)$

$E \to \lambda E(u)$

We may restrict $M(u) = M(Q)$

$$d_0(\Phi) := \operatorname{dist}_{H^1}(\Phi, \underbrace{\{e^{i\theta} Q\}_\theta}_{\mathcal{Q}})$$

Soln types:

$\mathcal{S}$: $\exists v$ st $i\dot{v} - \Delta v = 0$, $\|u(t) - v(t)\|_{H^1} \xrightarrow{t\to\infty} 0$

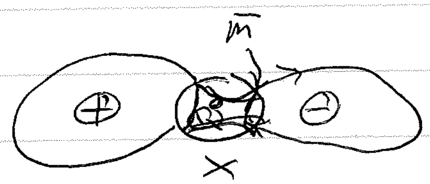
$\mathcal{B}$: $\exists T<0$ st $\|u(t)\|_{H^1} \to \infty$ as $t \searrow T$.

$\mathcal{J}$: $\overline{\lim_{t\to\infty}}\, d_0(u(t)) \lesssim \varepsilon$ $\quad (M(u) = M(Q))$

Main steps: ← neighbor of Q

- $\mathcal{H}(\varepsilon)\big|_{M(\varepsilon)=M(Q)} = X \sqcup \oplus \sqcup \ominus$

$\mathcal{H}(\varepsilon)\big|_{M(u)=M(Q)}$ $\quad X=\overline{X} \quad \oplus = \overline{\oplus}$



- u gets out of $X \Rightarrow$ it can't return to X (one-pass lemma)

- Staying $X/\oplus/\ominus \to \mathcal{J}/\mathcal{B}/\mathcal{S}$

- $\exists M \subset X$ a C^1 mfd codim $=1$
  $Q \subset M \subset \mathcal{J}$
  "Below M" $\Rightarrow$ ejected out of X
  "Above M" into $\oplus \rightrightarrows \begin{matrix}\mathcal{S}\\ \mathcal{B}\end{matrix}$

<u>Virial and Variational dichotomy</u> (Payne-Sattinger)

L^2 dilation $S^\tau \varphi := e^{3/2\,\tau}\varphi(e^\tau x)$

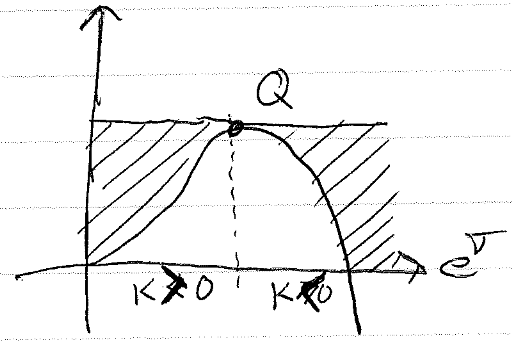
$$S' = x\cdot\nabla + 3/2$$

$$(NLS) \Leftrightarrow \dot{u} = iE'(u)$$

$$\partial_t \langle iu \,|\, S'u\rangle = -2\underbrace{\langle E'(u)\,|\,S'u\rangle}_{} \quad \partial_{\tau=0} E(S^\tau u) \,/\!/\, \text{"}K(u)$$

$$= \int_{\mathbb{R}^3} |\nabla u|^2 - \tfrac{3}{4}|u|^4$$

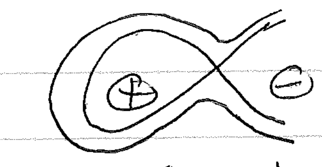
$E(S^\tau u)$



$$E(Q) = \inf_{M(\varphi)=M(Q)} \max_{\tau\in\mathbb{R}} E(S^\tau \varphi)$$

($\Leftarrow$ L^2-supercritical power)

H^1_r



Sign K

Assume $d_0(u) \geq \delta$

$\exists C > 0, \ \forall \delta > 0, \ \exists \varepsilon_v > 0$

$\exists K_v > 0$

$$J(u) := E(u) + M(u) < J(Q) + \varepsilon_v(\delta)^2$$

$$K(u) < -K_v(\delta)$$

$$K(u) > \min(K_v(\delta), C\|\nabla u\|_2^2)$$

Expansion around Q (Weinstein)

modulation analysis around Soliton

$u = e^{i\theta}(Q + v), \quad \|v\|_{H^1} \ll 1$

$$J(u) = J(Q) + \underbrace{\langle J'(Q)|v\rangle}_{=0 \text{ since } J'(Q)=0} + \frac{1}{2}\langle v|\overbrace{J''(Q)}^{\mathcal{L}}v\rangle - \underbrace{C(v)}_{O(\|v\|_{H^1}^3)}$$

$$\mathcal{L}v := L_1 v_1 + i L_2 v_2 \qquad \begin{cases} L_1 = -\Delta + 1 - 3Q^2 \\ L_2 = -\Delta + 1 - Q^2 \end{cases}$$

$$v = v_1 + i v_2$$

Thus

$$\text{(NLS)} \quad \dot{v} = i\mathcal{L}v - i\underbrace{C'(v)}_{O(v^2)} - i\underbrace{(\dot{\theta}+1)(Q+v)}_{O(v^2)}$$

$\partial_\theta, \partial_\omega$ on $Q \Rightarrow i\mathcal{L}iQ = 0, \quad i\mathcal{L}Q' = -iQ$

min-max of $S^\sigma \Rightarrow \exists \rho_\pm = \bar{\rho}_\mp \in H^1_r \quad i\mathcal{L}\rho_\pm = \pm\kappa\rho_\pm$

$\exists \kappa > 0$

$\dot{v} = i\mathcal{L}v \Leftarrow v = iQ, \ Q' - iQt, \ e^{\pm\kappa t}\rho_\pm$

Choice (θ, ω) $\quad \underbrace{M(u) = M(Q)}_{\Downarrow \ \langle iv|iQ\rangle = -M(v)}$ and $0 = \langle iv|Q'\rangle = \langle ie^{-i\theta}u|Q'\rangle$

$$0 = \langle i e^{-i\theta} u | Q' \rangle \Rightarrow \dot{\theta} + 1 = O(\|v\|_{H^1}^2)$$

Decompose $V = \lambda_+ \rho_+ + \lambda_- \rho_- + \gamma$ st $\langle i\gamma | \rho_\pm \rangle = 0$

$$= \lambda_1 \rho_1 + \lambda_2 \rho_2 + \gamma \qquad \rho_\pm = \underset{(Re)}{\rho_1} \pm i \underset{(Im)}{\rho_2}$$

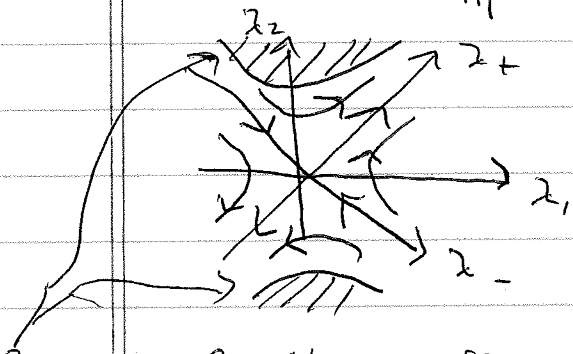
$$\lambda_\pm = \tfrac{1}{2}(\lambda_1 \pm \lambda_2$$

$$i\mathcal{L}v = \alpha \lambda_+ \rho_+ - \alpha \lambda_- \rho_- + i\mathcal{L}\gamma$$

$$J(u) = J(Q) \underbrace{- \alpha \lambda_+ \lambda_-}_{= \frac{\alpha}{4}(-\lambda_1^2 + \lambda_2^2)} + \underbrace{\tfrac{1}{2}\langle \mathcal{L}\gamma | \gamma \rangle}_{\sim \|\gamma\|_{H_1}^2} - C(v)$$

$$\|v\|_E^2 := \frac{\alpha}{4}(\lambda_1^2 + \lambda_2^2) + \tfrac{1}{2}\langle \mathcal{L}\gamma | \gamma \rangle = J(u) - J(Q) + \frac{\alpha}{2}\lambda_1^2 + C(v)$$

$$\sim \|v\|_{H_1}^2 \sim d_0(u)$$



energy constraint removes these regions.

$$\therefore \quad J(u) - J(Q) \ll d_0(u)^2 \ll 1 \implies$$

$$\|v\|_E \sim |\lambda_1|$$

(NLS) (MOD)

$$\Rightarrow \dot{\lambda}_\pm = \pm \alpha \lambda_\pm + O(\|v\|_E^2)$$

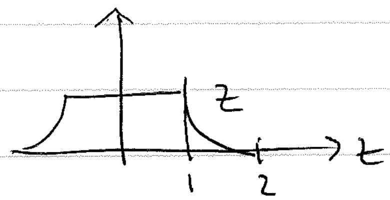
$$\partial_t \lambda_1^2 = 2\alpha \lambda_1 \lambda_2 + O(v^3)$$

$$\partial_t \lambda_1 \lambda_2 = 2\alpha^2(\lambda_1^2 + \lambda_2^2) + O(v^3)$$

$$\dot{\lambda}_1 = \alpha \lambda_2 + O(\|v\|_E^2)$$

$$\dot{\lambda}_2 = \alpha \lambda_1 + O(\|v\|_E^2)$$

Ejection Lem



$$d(u(0)) = \left(\int_{\mathbb{R}} \chi(t) \|v(t)\|_E^2 \, dt \right)^{1/2}$$

$$\Leftrightarrow d(u(t))^2 = \chi * \|v(t)\|_E^2$$
$$\sim d_0(u)^2$$
$$\partial_t^2 d(u(t))^2 = \chi * \kappa^2 (\lambda_1^2 + \lambda_2^2) + O(d_0^3)$$
$$J(u) - J(Q) \ll d(u)^2 \ll 1 \Rightarrow$$
$$d(u)^2 \sim \lambda_1^2 \sim \partial_t^2 d(u)^2$$

$$\|\gamma\|_E = \tfrac{1}{2} \langle \mathcal{L}\gamma | \gamma \rangle = J(u) - J(Q + \lambda_1 P_1 + \lambda_2 P_2) + O(\|\gamma\|_E \|v\|_E^2)$$
$$\Rightarrow \|\gamma\|_{L_t^\infty E} \lesssim \|\gamma(0)\|_E + \|v\|^2_{L_t^2 E + L_t^\infty E}$$

If $\underset{\text{const}}{C_*} (J(u) - J(Q)) \le d(u(0)) \ll 1$ and $\partial_t d(u(0)) \ge 0$

Then $d(u(t)) \nearrow \delta_*$ $(0 < \underset{\text{const}}{\delta_*} \ll 1)$ for $0 < t \le T_*$

$$d(u) \sim |\lambda_1| \sim e^{\kappa t} d(u(0))$$
$$|\lambda_-| + \|\gamma\| \lesssim d(u(0)) + d(u)^2$$

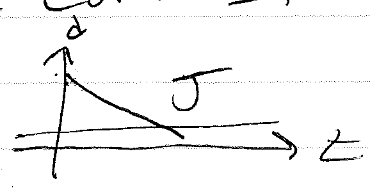
$$K(u) = -c_K \lambda_1 + O(d(u(0)) + d(u)^2)$$
$$c_K = -\tfrac{1}{2} \langle 2, Q | P_1 \rangle > 0 \quad \text{const}$$

Cor: If $\sup_{0<t<\infty} d(u(t)) < \delta_*$ Then $0 < \exists T \le \infty$

$0 < t < T$ $d(u(t)) \searrow$ (exp.)

$T \le t < \infty$ $d(u(t)) \lesssim J(u) - J(Q)$

$(u(0) \in J)$



$$\mathcal{H}(\varepsilon)\big|_{M(u)=M(0)} = X \sqcup \oplus \sqcup \ominus$$

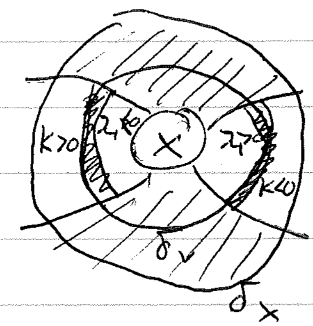
Take $0 < \delta_v \ll \delta_x$, $0 < \varepsilon \lesssim \varepsilon_v(\delta_v)$

$X = \{ d \le C_x \varepsilon \}$

Then $C_x \varepsilon < d \le \delta_x \Rightarrow \operatorname{sign} \lambda_1 = \text{const}$

$\delta_v \le d \Rightarrow \operatorname{sign} K = \text{const}$

$\pm = \operatorname{sign} K = -\operatorname{sign} \lambda_1$



even if K changes sign,

$$\int_0^{T_x} K(u)\,dt \sim \pm \delta_x$$