Invariant measures for nonlinear PDE

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The defocusing cubic NLS on $\mathbb{T}^2$

(NLS) $$\begin{cases} iu_t + \Delta u = |u|^2 u \\ u(x,0) = \phi \in H^s(\mathbb{T}^2) \end{cases}$$

$$\begin{aligned} \underline{\text{Mass:}} \quad & M(u(t)) &:=& \int |u(t,x)|^2\, dx \\ \underline{\text{Hamiltonian:}} \quad & H(u(t)) &:=& \frac{1}{2}\int |\nabla u(t,x)|^2\, dx + \frac{1}{4}\int |u(t,x)|^4 dx \end{aligned}$$

are both constant in time.

- The equation is L^2 critical ($s_c = 0$).
- Bourgain (93') proved LWP for $s > 0$ and GWP in $H^1(\mathbb{T}^2)$.
  - Recall that in the first lecture we discussed the ϵ-loss of derivatives in the $L^4_{xt}(\mathbb{T}^2 \times \mathbb{T})$ Strichartz estimate for the linear evolution. This accounts for the need of $s > 0$ to close the fixed point argument.

Some Issues ...

We are interested in the existence and invariance of the Gibbs measure for the defocusing cubic NLS on $\mathbb{T}^2$, *formally* given by

$$
\begin{aligned}
\text{"}\, d\mu &= Z^{-1} e^{-H(\phi)} \prod_{x \in \mathbb{T}^2} d\phi(x) \quad \text{"} \\
&= \text{"} \quad Z^{-1} e^{-\int |\phi|^4\,dx} \underbrace{e^{-\frac{1}{2}\int |\nabla \phi|^2\,dx} \prod_{x \in \mathbb{T}^2} d\phi(x)}_{\text{Gaussian measure } d\rho} \quad \text{"} \\
&= \text{"} \quad e^{-\int |\phi|^4\,dx}\, d\rho \quad \text{"}
\end{aligned}
$$

and in the almost sure global well posedness on its support.

Recall from the first lecture that in 2D, the Gaussian measure ρ on $H^s(\mathbb{T}^2)$ is countably additive if and only if $B_s := (1 - \Delta)^{s-1}$ on $\mathbb{T}^2$ is of trace class; i.e. if and only if $s < 0$.

Recall also that in 2D, ρ yields for ϕ the distribution of a random (Fourier) series

$$\phi = \phi^\omega = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{in\cdot x}.$$

which defines a.s. a distribution -not a function- in $H^s(\mathbb{T})$, $s < 0$.

Hence, **unlike the 1D case**, in 2D for the typical ϕ the expression $e^{-\int_{\mathbb{T}^2} |\phi|^4\,dx}$ is unbounded a.s.; i.e

$$\lim_{N\to\infty} \int_{\mathbb{T}^2} |P_N(\phi^\omega)|^4\,dx \;=\; \infty \qquad \text{a.s. in } \omega$$

where as before $P_N(\phi^\omega) = \sum_{|n|\leq N} \frac{g_n(\omega}{\sqrt{1+|n|^2}} e^{in\cdot x} =: \phi_N^\omega$.

To overcome this problem Bourgain considers the *Wick ordering*[1] of $|\phi_N|^4$.

[1] as in QFT

Let

$$a_N := \mathbb{E}(|\phi_N^\omega|^2) \sim \sum_{|n|\leq N} \frac{1}{1+|n|^2} \sim \log N \quad \text{(2D)}.$$

After renormalizing by a_N the Wick ordering of $|\phi_N|^4$ (complex) is given by:

$$: |\phi_N|^4 : = |\phi_N|^4 - 4a_N|\phi_N|^2 + 2a_N^2.$$

Proposition (Bourgain 96')

(1) $\int_{\mathbb{T}^2} : |\phi_N^\omega|^4 : \, dx$ *converges a.s. in* ω *to a finite limit as* $N \to \infty$.

(2) The measures $d\mu_N := e^{-\int :|\phi_N|^4: dx} \, d\rho_N$ *converge to a weighted Wiener measure with density in* $L^r(d\rho), r < \infty$. *Call this measure* μ

Very roughly: In the real case, Wick ordering consists in associating to a monomial x^n a Hermite polynomial $He_n(x)$ obtained by orthogonalization of the monomials w.r.t Gaussian measure on $\mathbb{R}$.

$$He_n(x) := 2^{-n/2} H_n(\frac{x}{\sqrt{2}}),$$

where

$$H_n(x) := \sum_{m=0}^{[\frac{n}{2}]} \frac{(-1)^m n!}{m! 2^m (n-2m)!} \, x^{n-2m}$$

Note the recursion relation

$$He_{n+1}(x) = xHe_n(x) - nHe_{n-1}(x).$$

So for example we have

$$He_0(x) = 1, He_1(x) = x,\ He_2(x) = x^2 - 1,\ He_3(x) = x^3 - 3x,\ He_4(x) = x^4 - 6x^2 + 3, \text{ etc.}$$

In the complex case and the notation above

$$: |\phi_N|^{2k} := \ a_N^k H_{2k}(\frac{\phi_N}{\sqrt{a_N}})$$

(c.f. Bourgain's IAS/Park City Lecture Notes Vol 5, 1999).

The Wick ordering above leads to a modified Hamiltonian:

$$\begin{aligned}\mathcal{H}_N(\phi_N) &:= \int |\nabla \phi_N|^2 dx + \frac{1}{2}\int_{\mathbb{T}^2} :|\phi_N|^4: \, dx \\ &= \int |\nabla \phi_N|^2 dx + \frac{1}{2}\int_{\mathbb{T}^2} |\phi_N|^4 \, dx - 2a_N \int |\phi_N|^2 + a_N^2,\end{aligned}$$

whence we obtain the Wick ordered NLS equation (WNLS):

$$\begin{aligned} i\,\partial_t u_N &= \frac{\partial \mathcal{H}_N}{\partial \overline{u_N}} \\ i\,\partial_t u_N &= -\Delta u_N + P_N(|u_N|^2 u_N) - 2a_N u_N \end{aligned}$$

or equivalently

$$i\,\partial_t u_N + \Delta u_N + 2(a_N - \int |u_N|^2 \, dx) u_N - P_N\big(u_N |u_N|^2 - 2u_N \int |u_N|^2 \, dx\big) = 0$$

Using the L^2 conservation,

$$\int |u_N|^2 \, dx - a_N \; = \int : |\phi_N|^2 : \, dx \; = \underbrace{c_N(\omega)}_{\text{independent of time}} \longrightarrow_{N\to\infty} c(\omega), \text{ a.s. in } \omega,$$

whence we get

$$i\,\partial_t u_N \, + \, \Delta u_N + 2 c_N u_N - P_N\Big(u_N |u_N|^2 \, - \, 2 u_N \int |u_N|^2 \, dx\Big) = 0$$

and the linear term maybe simply removed by letting $v_N := e^{2ic_N t} u_N$ satisfying

(FWNLS) $$i\,\partial_t v_N \, + \, \Delta v_N - P_N\Big(v_N |v_N|^2 \, - \, 2 v_N \int |v_N|^2 \, dx\Big) = 0,$$

which is the truncated or finite dimensional approximation to:

(WNLS) $$i\,\partial_t v \, + \, \Delta v - \Big(v|v|^2 \, - \, 2v \int |v|^2 \, dx\Big) = 0,$$

the *Wick ordered cubic NLS* equation.

The measures $d\mu_N = e^{-\mathcal{H}_N(\phi_N)} \Pi d^2 \phi_N = e^{-\int :|\phi_N|^4 : dx} d\rho_N$ are invariant under the flow of (FWNLS).

The weighted Wiener measure μ to which the invariant measures μ_N converge -according to the Proposition above- **should be** the invariant Gibbs measure associated to (WNLS). To conclude this –just as we have seen for the 1D quintic NLS– the main two outstanding issues are:

1. Some form of local well-posedness **below** $L^2(\mathbb{T}^2)$ for (WNLS).
   - Recall ρ necessitates the flow to be well-posed in $H^s(\mathbb{T}^2)$, $s < 0$.
   - $s < 0$ corresponds to the supercritical regime where even some small data could -in principle- lead to 'bad' behavior in short times.
2. An approximation lemma (uniform convergence of v_N to v.)
   - This approximation lemma is similar but a bit more delicate than in 1D because of the form of the solution in the a.s. LWP result below. (c.f. A.N-Staffilani, arXiv:1507.07320 [Prop 3.5]).

Then relying on (2) above and the invariance of μ_N one can prove the **almost sure global well posedness for WNLS in $H^{-\varepsilon}(\mathbb{T}^2)$** (as in the 1D case) and the **invariance of the Gibbs measure μ under the WNLS flow.** (Bourgain 96').

- The main issue to address then is (1) above since we **do not have** a deterministic LWP in $H^s(\mathbb{T}^2)$, $s < 0$ in place (as it was the case in 1D).
  - At present, we do not even have a deterministic well-posedness in $L^2(\mathbb{T}^2)$!

The main point however is that one only needs local well-posedness in the support of the measure. That is, it is enough to prove almost sure local well-posedness in $H^s(\mathbb{T}^2)$, $s < 0$ of WNLS.

Theorem (a.s LWP **Bourgain**(96'))

The Cauchy initial value problem

(WNLS)
$$\begin{cases} iv_t + \Delta v = |v|^2 v \; - \; 2(\int |v|^2 dx)\, v \\ v(x,0) = \phi^\omega = \sum_n \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{ix\cdot n}, \quad x \in \mathbb{T}^2, \end{cases}$$

is locally well-posed on a time interval $[0,\tau]$ *except for* ω *in a set* Ω_τ^c *of measure at most* $e^{-\frac{1}{\tau}}$

The solution v *is the distributional limit of* v^N*, the solution to (FWNLS) with initial data* $v^N(0) = P_N(\phi^\omega)$.

Furthermore, almost surely in ω *the nonlinear part*

$$w := \; v - S(t)\phi^\omega \in C([0,\tau]; H^\alpha(\mathbb{T}^2)), \; \alpha > 0.$$

i.e. is **smoother** *than the linear part.*

Here $S(t)\phi^\omega$ is the solution to the linear problem with data ϕ^ω.

Randomization does not improve regularity in terms of derivatives!
The initial data,

$$\phi^{\omega}(x) = \sum \frac{g_n(\omega)}{|n|} e^{i\langle x,n\rangle},$$

defines almost surely in ω a function in $H^{-\epsilon}$; **but not** in H^s, $s \geq 0$. In other words, it is as regular as

$$\phi(x) = \sum \frac{1}{|n|} e^{i\langle x,n\rangle}.$$

- Why does randomization help ?
- **Key Point:** The linear flow $S(t)\phi^{\omega}(x)$ of **rough but randomized data** enjoys *almost surely* improved L^p bounds.
  - Results of Rademacher, Kolmogorov, Paley and Zygmund show that **random series** enjoy better L^p bounds than deterministic ones.
  - Randomness has classically been introduced into Fourier series as a tool for answering deterministic questions (Paley and Zygmund 30's)
  - Phenomena akin to how Kintchine inequality is used in Littlewood-Paley theory.

Classical Example

Consider *Rademacher Series* :

$$f(y) := \sum_{n=0}^{\infty} a_n\, r_n(y) \qquad\qquad y \in [0,1), \quad a_n \in \mathbb{C}$$

where

$$r_n(y) := \mathrm{sign}\, \sin(2^{n+1}\pi\, y)$$

We have:

- If $a_n \in \ell^2$ the sum $f(y)$ converges a.e.

Classical Theorem

If $a_n \in \ell^2$ then the sum $f(y)$ belongs to $L^p([0,1))$ for all $p \geq 2$. More precisely,

$$\Big(\int_0^1 |f|^p\, dy \,\Big)^{1/p} \approx_p \| \, a_n \|_{\ell^2}$$

Ex. $a_n = c_n e^{i\, n\theta}, \quad c_n \in \ell^2$.

- These a.s. improved L^p bounds on the linear evolution in turn yield improved nonlinear estimates *almost surely* in the analysis of

$$w(t,x) = v(t,x) - S(t)\phi^{\omega}(x),$$

where v is the solution of the equation at hand and as a consequence w **solves a difference equation:**

(DE)
$$\begin{cases} iw_t + \Delta w = \mathcal{N}(w + S(t)\phi^{\omega}) \\ w(x,0) = 0 \end{cases}$$

where $\mathcal{N}(f) = (|f|^2 f - 2f \int |f|^2)$

Remark (Important)

The difference equation that w solves is <u>not</u> back to merely being at a 'smoother' level but rather it is a hybrid equation with nonlinearity $=$
$=$ supercritical (but random) $+$ deterministic (smoother).

- Randomization techniques have now been used in several contexts and regimes to **improve the LWP almost surely**. How to pass from LWP to global is a separate issue which depends on the equation, the dimension, and the regime (invariant measures, energy methods, probabilistic adaptations of Bourgain's high-low/I-method, etc.)

- Schrödinger Equations: Bourgain, Tzvetkov, Thomann-Tzvetkov, A.N.-Oh-Rey-Bellet-Staffilani, A.N.-Rey-Bellet- Sheffield-Staffilani, Colliander-Oh, Burq-Thomann-Tzevtkov, Y. Deng, Burq-Lebeau, Bourgain-Bulut, A.N.- Staffilani, Poiret-Robert-Thomann, Bényi- Oh- Pocovnicu (conditional), . . .

- KdV Equations: Bourgain, Oh, Richards.

- NLW/NLKG Equations: Burq-Tzvetkov, de Suzzoni, Bourgain-Bulut, Luehrmann-Mendelson, S. Xu, Pocovnicu, Oh-Pocovincu, Mendelson.

- Benjamin-Ono Equations: Y. Deng, Tzvetkov-Visciglia. and Y. Deng-Tzvetkov-Visciglia.

- Navier-Stokes Equations: A.N.-Pavlovic-Staffilani: infinite 'energy' global (weak) sols in $\mathbb{T}^2, \mathbb{T}^3$, global energy bounds, uniqueness in $\mathbb{T}^2$. Also work by Deng-Cui and Zhang-Fang

Heart of the matter. The difference equation

- One proceeds via a fixed point argument on a suitable Banach space $X^s \subset C([0,\tau]; H^\alpha(\mathbb{T}^2))$.

- To set up a contraction, the main estimate one needs is essentially:

$$\| \int_0^t S(t-t')\mathcal{N}(w + S(t)\phi^\omega)\, dt' \|_{X^s} \lesssim \tau^\gamma (1 + \|w\|_{X^s}^3)$$

  for some $\gamma > 0$ and $\omega \in \Omega_\tau$

- Recall ϕ^ω belongs only $H^{-\varepsilon}(\mathbb{T}^2)$.

- The heart of the matter is to prove suitable estimates for $\mathcal{N}(w + S(t)\phi^\omega)$.

- $\mathcal{N}(w + S(t)\phi^\omega)$ consists essentially of cubic terms which may be all random (R), all deterministic (D), or mixed.

- The Wick ordering of the Hamiltonian crucially removed certain bad *resonant* frequencies!

Large Deviation-type result

Let k be the number of random terms in the multilinear estimate at hand.

Let $d \geq 1$ and $c(n_1, \ldots, n_k) \in \mathbb{C}$. Let $\{g_n\}_{1 \leq n \leq d} \in \mathcal{N}_{\mathbb{C}}(0,1)$ be complex centered L^2 normalized independent Gaussians. For $k \geq 1$ denote by

$$A(k,d) := \{(n_1, \ldots, n_k) \in \{1, \ldots, d\}^k,\ n_1 \leq \cdots \leq n_k\}$$

and

$$F_k(\omega) = \sum_{A(k,d)} c(n_1, \ldots, n_k) g_{n_1}(\omega) \ldots g_{n_k}(\omega).$$

Then for $p \geq 2$

$$\|F_k\|_{L^p(\Omega)} \lesssim \sqrt{k+1}(p-1)^{\frac{k}{2}} \|F_k\|_{L^2(\Omega)}.$$

If $L := \Delta - x \cdot \nabla$, the Hartree-Fock operator defined as the self adjoint realization on $L^2(\mathbb{R}^d, \exp(-|x|^2/2)dx)$, Dom $= \{u\ :\ u(x) = e^{|x|^2/4} v(x), \quad v \in H^{\alpha,\beta}, |\alpha| + |\beta| \leq 2\}$. The hyper-contractivity property of the Ornstein-Uhlenbeck semigroup e^{-tL} gives L^p-L^q estimates for the heat flow. Write $g_n = h_n + i\ell_n$ where $\{h_1, \ldots, h_d, \ell_1, \ldots \ell_d\} \in \mathcal{N}_{\mathbb{R}}(0,1)$ are real centered independent Gaussian random variables with the same variance and re-express as Hermite polynomial, hence an eigenvector for semigroup (c.f. Tzvetkov)

As a consequence from Chebyshev's inequality for every $\lambda > 0$,

$$\mathbb{P}(\{\omega \,:\, |F_k(\omega)| > \lambda\}) \leq \exp\left(\frac{-C\,\lambda^{\frac{2}{k}}}{\|F_k(\omega)\|_{L^2(\Omega)}^{\frac{2}{k}}}\right).$$

Given $\tau > 0$, the large deviation result above with -say -

$$\lambda = \tau^{-\frac{3}{2}}\|F_k(\omega)\|_{L^2(\Omega)}$$

so that in a set Ω_τ with $\mathbb{P}(\Omega_\tau^c) < e^{-\frac{1}{\tau}}$ we can replace $|F_k(\omega)|^2$ by $\|F_k(\omega)\|_{L^2(\Omega)}^2$.

An Explicit Estimate

Let us assume, that $N_1 \gg N_2 \geq N_3$ are dyadic numbers, that have been fixed.

Let us consider the all random case $R_1 R_2 \overline{R}_3$ in the nonlinear term; ie. R_j is the linear evolution of the random data.

Thanks to the Wick ordering we know that $n_1, n_2 \neq n_3$ where n_j is the spatial frequency of R_j.

Let us also assume that we have perform a LP decomposition and that R_j is frequency localized to N_j.

After further decomposing the frequency annulus of R_1 by boxes C of sidelength N_2 and using LP again, we need to estimate:

$$\| P_C R_1 R_2 \overline{R}_3 \|_{L^2_{xt}}$$

for $\omega \in \Omega_\tau$. We would like to obtain decay in N_1 so as to absorb a derivative of order $\alpha > 0$.

By Plancherel we reduce the estimate to

$$\|P_C R_1 R_2 \overline{R}_3\|_{L^2}^2 = \sum_{m,n\in\mathbb{Z}^2} \left| \sum_{\substack{n=n_1+n_2-n_3 \\ n_1\neq n_3; n_2\neq n_3,\ n_1\in C \\ m=|n_1|^2+|n_2|^2-|n_3|^2}} \frac{\overline{g}_{n_1}(\omega)}{|n_1|} \frac{\overline{g}_{n_2}(\omega)}{|n_2|} \frac{g_{n_3}(\omega)}{|n_3|} \right|^2 = |F_3(\omega)|^2$$

There are two cases

- **Case** A_0**:** The frequencies n_i, $i = 1, 2, 3$ are all different from each other.
- **Case** A_1**:** $n_1 = n_2$.

Case A_0: We first remark that the **variation for the time frequency** m is

$$\Delta m \sim N_1 N_2.$$

Then we use the **large deviation-type** result with

$$\lambda \sim \tau^{-\frac{3}{2}} \|F_3(\omega)\|_{L^2(\Omega)}$$

so that in a set Ω_τ with $\mathbb{P}(\Omega_\tau^c) < e^{-\frac{1}{\tau}}$ we can replace $|F_3(\omega)|^2$ by $\|F_3(\omega)\|_{L^2(\Omega)}^2$.

Then we write

$$\|F_3(\omega)\|^2_{L^2(\Omega)} = \sum_{S_{(n,m)}} \sum_{S_{(n,m)}} \chi_C(n_1) \prod_{i=1}^{3} \frac{1}{|n_i|} \chi_C(n_1') \prod_{j=1}^{3} \frac{1}{|n_j'|}$$
$$\times \int_\Omega \overline{g}_{n_1}(\omega)\overline{g}_{n_2}(\omega) g_{n_3}(\omega) \overline{g}_{n_1'}(\omega)\overline{g}_{n_2'}(\omega) g_{n_3'}(\omega)\, dp(\omega)$$

where $S_{(n,m)}$ is the set of triplets

$$\{(n_1, n_2, n_3) : n = n_1 + n_2 - n_3,\ n_1, n_2 \neq n_3,\ m = |n_1|^2 + |n_2|^2 - |n_3|^2\}.$$

Using the independence and normalization of $g_n(\omega)$, everything contracts to

$$\|F_3(\omega)\|^2_{L^2(\Omega)} = \sum_{S_{(n,m)}} \chi_C(n_1) \prod_{i=1}^{3} \frac{1}{|n_i|^3}$$

and we proceed to obtain

$$\|P_C R_1 R_2 \overline{R}_3\|^2_{L^2} \lesssim N_1 N_2 \sum_n |F_3(\omega)|^2 \lesssim \tau^{-\frac{3}{2}} N_1 N_2 N_1^{-2} N_2^{-2} N_3^{-2} \sup_m \# S(m)$$

where

$$S_m := \{(n, n_1, n_2, n_3) \,/\, n = n_1 + n_2 - n_3 \,;\, m = |n_1|^2 + |n_2|^2 - |n_3|^2, n_1 \in C\}.$$

Since

$$\# S_m \lesssim N_3^2 N_2^2 N_1^{\epsilon}$$

we then obtain in Case A_0 the bound:

$$\|P_C \bar{R}_1 \bar{R}_2 R_3\|^2_{L^2} \lesssim \tau^{-\frac{3}{2}} N_1^{-1} N_2.$$

For **Case** A_1**:** now we have $n_1 = n_2$ and,

$$\|P_C R_1 R_2 \overline{R}_3\|_{L^2}^2 := \sum_{m,n\in\mathbb{Z}^2} \left| \sum_{\substack{n=2n_1-n_3 \\ n_1\neq n_3; \\ m=2|n_1|^2-|n_3|^2}} \frac{(g_{n_1}(\omega))^2}{|n_1|^2} \frac{\overline{g}_{n_3}(\omega)}{|n_3|} \right|^2$$

$$= \sum_{m,n\in\mathbb{Z}^2} \left| \sum_{\substack{n_1\neq n; \\ m=2|n_1|^2-|2n_1-n|^2}} \frac{(g_{n_1}(\omega))^2}{|n_1|^2} \frac{\overline{g}_{2n_1-n}(\omega)}{|2n_1-n|} \right|^2$$

We can continue in this case by Cauchy Schwarz to obtain:

$$\sum_{m,n\in\mathbb{Z}^3}\left|\sum_{\substack{n_1\neq n;\\ m=2|n_1|^2-|2n_1-n|^2}}\frac{(g_{n_1}(\omega))^2}{|n_1|^2}\frac{\overline{g}_{2n_1-n}(\omega)}{|2n_1-n|}\right|^2$$

$$\lesssim \sum_{m,n\in\mathbb{Z}}\#\tilde{S}_{(n,m)}\sum_{n_1;\ m=2|n_1|^2-|2n_1-n|^2}\frac{|g_{n_1}(\omega)|^4}{|n_1|^4}\frac{|\overline{g}_{2n_1-n}(\omega)|^2}{|2n_1-n|^2}$$

where

$$\tilde{S}_{(n,m)}=\{n_1\ /\ m=2|n_1|^2-|2n_1-n|^2\}.$$

Since $\#\tilde{S}_{(n,m)}\lesssim N_1^{\epsilon}$ and we can show that $|g_{n_1}(\omega)|\lesssim N_1^{\varepsilon}$, we obtain a much better decay in this case than in Case A_0.

Final Remarks

- Brydges and Slade showed it is not possible to carry over the canonical construction of Gibbs measures for the focusing cubic NLS on $\mathbb{T}^2$.
- Invariant measures for Hamiltonian PDE in higher dimensions remain a challenge ....
- Little is known about ergodicity of (nonlinear) Hamiltonian PDE's.
  - Lebowitz and Lanford 74' (eg. linear wave equation and more general linear PDE)
  - Jaksic and Pillet 90' (PDE coupled to a finite dimensional ODE)
  - McKean 95' (hyperbolic sine-Gordon)