Kenji Nakanishi   Lecture 3

(NLS) $i\dot{u} - \Delta u = |u|^2 u$

$u(t,|x|) : \mathbb{R}^{1+3} \to \mathbb{C}$, $u(0) \in H^1_r$

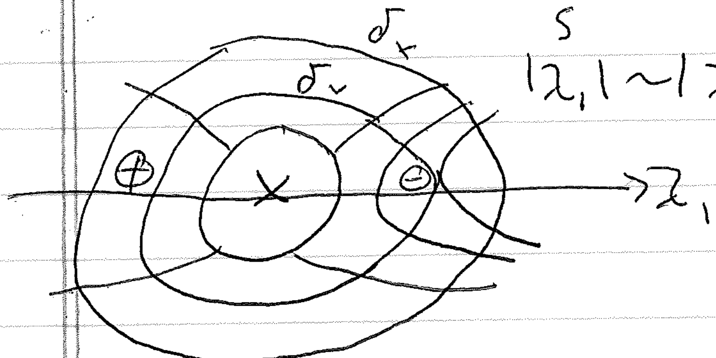
$M(u) = M(Q)$, $E(u) < E(Q) + \varepsilon^2$

$-\Delta Q + Q = Q^3$ $0 < \varepsilon \ll 1$

$C_*^2 (J(u) - J(Q)) \le d(u)^2 \ll 1$

$\partial_t d(u(0)) \ge 0$

$\Rightarrow d(u(t)) \nearrow \delta_*$ $K = -C\lambda_1 \ne 0$



One-Pass Lem

If $C_* \varepsilon \le d(u(0)) = \delta \ll \delta_*$. $\partial_t d(u(0)) \ge 0$

$\Rightarrow \forall t > 0 \quad d(u(t)) > \delta$

$\because$ local virial $A(t) := \langle iu \mid \chi(\delta|x|) S' u \rangle$

$\oplus$ $\chi(r) = \frac{1}{1+r}$ $-\dot{A} \le 2K(\chi u)$ + error

If at some $t = T > 0$ $d(u(T)) = \delta$,

$\int_0^t K(\chi u)\,dt \gtrsim \delta_*$ $|A(0)| + |A(T)| \lesssim \delta$

Mfd in X separating S and B (Bates-Jones)

$u = e^{i\Theta}(Q + v)$

$\hookrightarrow \lambda_+ \rho_+ + \lambda_- \rho_- + \gamma$

Find $\lambda_+(0)$ st u is not ejected

for given $(\lambda_-(0), \gamma(0))$

$0 = \langle i\gamma | \rho_\pm \rangle = \langle i\gamma | Q' \rangle = \langle i\gamma | iQ \rangle$

$M(u) \neq M(Q) \longrightarrow \langle iv | iQ \rangle = 0 \Rightarrow$

Ejection Lemma.

If $|\lambda_-(0)| + \|\gamma(0)\|_E \ll |\lambda_+(0)| \ll \delta_X$

Then $\Rightarrow$ Ejected into $\pm$ $\Rightarrow$ stable $\Rightarrow$ for initial perturb

Corresponding $\{\lambda_+(0)\} \subset \mathbb{R}$

are open non-empty

disjoint

$\Downarrow$

$\exists \lambda_+(0)$ st u is not ejected.

Unique and Lipschitz

$$\left.\begin{array}{l} u^0(0), u^1(0) \in J \\ \text{as above} \\ d(u^j(0)) \leq \delta \end{array}\right\} \Rightarrow |\lambda_+^0(0) - \lambda_+^1(0)| \lesssim$$

$$\sqrt{\delta}(|\lambda_-^0(0) - \lambda_-^1(0)| + \|\gamma^0(0) - \gamma^1(0)\|_E)$$

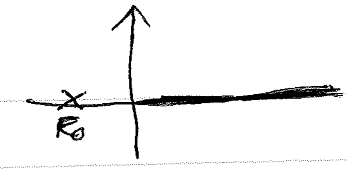
Applying the same to difference quotient $\rightarrow C^1$

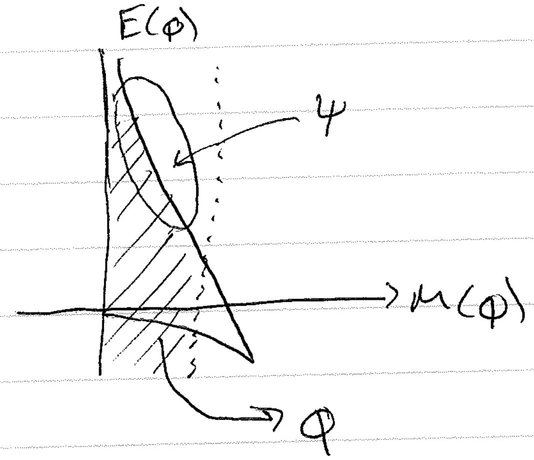
$$M_0 = \{e^{i\Theta}(Q + G(\lambda_-(0), \gamma(0))\rho_+ + \lambda_-(0)\rho_- + \gamma(0)) \mid \underset{\circledast}{|\lambda_-(0)| + \|\gamma(0)\| \ll 1}$$

$$\mathcal{H}(\varepsilon) \cap J \subset \{u(0) \in H^1_r \mid \exists T > 0, \exists \varphi \in M_0, u(T) = w^{1/2}\varphi(w^{1/2}x)$$

$\exists w > 0$

Idea for $V \neq 0$ $\quad H = -\Delta + V$





<u>difficulty by V</u> Global Dispersion

- broken virial $K(u)$ changes sign ϕ
- Scattering for $u - \Phi[z(t)]$ poor control on z

Scattering to Φ (Kenig-Merle)

- Find a <u>minimal</u> global sol $\notin S$ $\Updownarrow$ $u(t) - \Phi[z(t)] \notin L^4_t L^6_x = ST$

$\Downarrow$ irreducible

$\Downarrow$ precompact trajectory in H^1_r

Virial $\Rightarrow$ monotonicity $\# \leftarrow$ contradiction

$u(t) = \Phi[z(t)] + \eta(t) \qquad \langle i\eta \mid \Phi'[z] \rangle = 0$

$\Uparrow$

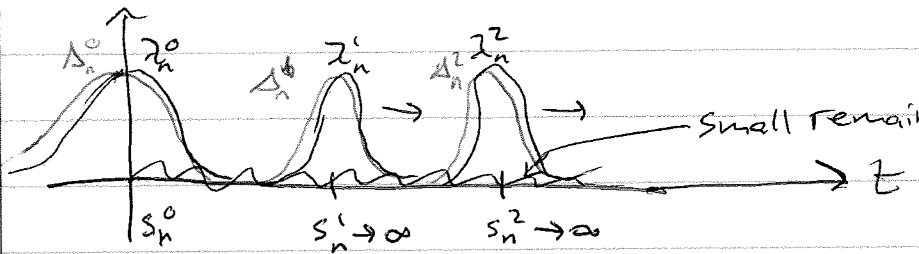
$M(u) \ll 1 \qquad \xi := P_c \eta = \eta - \Phi_0 \langle \Phi_0 \mid \eta \rangle$

(NLS) $\Leftrightarrow$ eq (z, ξ) $\quad S \Leftrightarrow \|\xi\|_{L^4_t L^6_x} < \infty$

Unique soln, $(E(u_n), M(u_n)) \rightarrow (E_*, M_*)$: minimal

$\|\xi_n\|_{ST} \rightarrow \infty$

<u>Loss of cpt.</u> $\Leftarrow$ time translation



ζ_n : Linearized Soln, $\xi_n(0) = \zeta_n(0)$

$\to \{s_n^0\}, \{s_n^1\}, \{s_n^2\} \ldots \qquad |s_n^j - s_n^k| \to \infty$

$\{s_n^0\} \equiv 0$, $\{s_n^1\} \to \infty$, $\{s_n^2\} \to \infty$

$$\zeta_n = \sum_{j=0}^{J-1} \lambda_n^j + \gamma_n^J$$

Let $(z_\infty^j, \xi_\infty^j) = \lim_{n\to\infty} (z_n, \xi_n)(t + s_n^j)$ (weak)

$\Downarrow$

soln of eq(z, ξ)

$$\Delta_n^j(t) = \xi_\infty^j(t - s_n^j)$$

Let Γ_n^J be sol of eq(z_n, Γ_n^J)

$$\Gamma_n^J(0) = \gamma_n^J(0)$$

If all Δ_n^j are scattering as $t \to \infty$ then

$$\xi_n = \sum \Delta_n^j + \Gamma_n^J$$

$\Downarrow$ $\|\xi_n\|_{ST} \le C < \infty$

$\Rightarrow$ at least (most) one profile does not scatter

$\Delta_n^j \to$ minimal sol $\notin S$

Δ_n^j can't be defined by wave operator

$\Uparrow$ poor control on z

$|z(t)| \to z_*$

One-pass lemma

sign $K(\chi u)$ may change

may approach Φ

$\leftarrow$ Kenig-Merle $\rightsquigarrow$ precpt

Chatterjee "a.s." scattering to grd st

$i\dot{u} - \Delta u = |u|^{p+1} u$

$1 < p < 1 + 4/d$: L^2 subcritical

$-\Delta Q + Q = Q^p$

Q "a.s." SUB

L^2-supercritical