Diffusive limits for stochastic kinetic equations

Arnaud Debussche

Ecole Normale Supérieure de Rennes.
Joint work with S. De Moor (ENS Rennes) and J. Vovelle (Lyon 1).

MSRI, Berkeley
October 19th, 2015

Kinetic models

Many physical systems are described by a kinetic equation:

$$\partial_t f + a(v) \cdot \nabla_x f = Q(f),$$

- $v \in V$ represents the various degrees of freedom of a particle, $a(v)$ is its velocity (often $a(v) = v$).
- $f(x, v)$ is the distribution function of the particles with degrees of freedom v at position $x \in \mathbb{T}^N$ (in this talk).
- V is endowed with a probability measure μ and the averaged velocity is zero : $\bar{a} = \int_V a(v) d\mu = 0$.
- Q accounts for the interaction between particles or between a particle and the medium.
- In general, it has a family of equilibrium F such that: $Q(f) = 0$ iff $f = \bar{f} F = \left(\int_V f d\mu\right) F$ with $F > 0$, $\bar{F} = 1$.
- Often, a small parameter ε is present in the equation and, after rescaling, the following equation is obtained:

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} Q(f^\varepsilon),$$

Radiative transfer and Rosseland approximation

- $$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}) L f^\varepsilon,$$

  with $L(f) = \bar{f} F - f$ describes the interaction between a surrounding continuous medium and a flux of photons radiating through it in the absence of hydrodynamical motion.
- The unknown $f^\varepsilon(t, x, v)$ then stands for a distribution function of photons having position x and velocity v at time t.
- The function σ is the opacity of the matter.

Radiative transfer and Rosseland approximation

- $$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}) L f^\varepsilon,$$

  with $L(f) = \bar{f} F - f$ describes the interaction between a surrounding continuous medium and a flux of photons radiating through it in the absence of hydrodynamical motion.
- The unknown $f^\varepsilon(t, x, v)$ then stands for a distribution function of photons having position x and velocity v at time t.
- The function σ is the opacity of the matter.
- When the surrounding medium becomes very large compared to the mean free paths ε of photons, f^ε is known to behave like ρ the solution of the Rosseland equation

  $$\partial_t \rho - \mathrm{div}_x(\sigma(\rho)^{-1} K \nabla_x \rho) \; = \; 0, \qquad (t, x) \in [0, T] \times \mathbb{T}^N.$$

  with $K := \int_V a(v) \otimes a(v)\, dv$. This is called the Rosseland approximation. (Bardos, Golse, Perthame, Sentis)

Deterministic equation, diffusive limit, $F = 1$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}) L(f^\varepsilon), \quad L(f) = \bar{f} - f.$$

Hilbert expansion (formal): $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$

Deterministic equation, diffusive limit, $F = 1$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}) L(f^\varepsilon), \quad L(f) = \bar{f} - f.$$

Hilbert expansion (formal): $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$

$\rightsquigarrow$ order -2 : $Lf_0 = \bar{f}_0 - f_0 = 0$ and $f_0 = \bar{f}_0 = \rho$.

(We assume $0 < \sigma_* \leq \sigma(\rho) \leq \sigma^*,\ \rho \in \mathbb{R}$).

Deterministic equation, diffusive limit, $F = 1$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}) L(f^\varepsilon), \quad L(f) = \bar{f} - f.$$

Hilbert expansion (formal): $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$

$\rightsquigarrow$ order -2 : $Lf_0 = \bar{f}_0 - f_0 = 0$ and $f_0 = \bar{f}_0 = \rho$.

(We assume $0 < \sigma_* \leq \sigma(\rho) \leq \sigma^*,\ \rho \in \mathbb{R}$).

$\rightsquigarrow$ order -1 : $a(v) \cdot \nabla_x \rho = \sigma(\rho) L(f_1)$.

Deterministic equation, diffusive limit, $F = 1$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}) L(f^\varepsilon), \quad L(f) = \bar{f} - f.$$

Hilbert expansion (formal): $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$

$\rightsquigarrow$ order -2 : $L f_0 = \bar{f}_0 - f_0 = 0$ and $f_0 = \bar{f}_0 = \rho$.

(We assume $0 < \sigma_* \leq \sigma(\rho) \leq \sigma^*, \ \rho \in \mathbb{R}$).

$\rightsquigarrow$ order -1 : $a(v) \cdot \nabla_x \rho = \sigma(\rho) L(f_1)$.

The equation

$$L(g) = \bar{g} - g = \int_V g d\mu - g = h$$

can be solved iff $\int_V h d\mu = 0$ and in this case, we can take

$$g = -h.$$

Recall that $\int_V a(v) d\mu = 0 \to f_1 = -\sigma(\rho)^{-1} a(v) \cdot \nabla_x \rho$

Deterministic equation, diffusive limit, $F = 1$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}^\varepsilon) L f^\varepsilon, \quad L(f) = \bar{f} - f.$$

Hilbert expansion (formal): $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$

$\rightsquigarrow$ order -2 : $L(f_0) = 0$ and $f_0 = \int_V f_0 d\mu = \rho$.

$\rightsquigarrow$ order -1 : $a(v) \cdot \nabla_x \rho = L(f_1) \longrightarrow f_1 = -\sigma(\rho)^{-1} a(v) \cdot \nabla_x \rho$.

$\rightsquigarrow$ order 0 : $\partial_t \rho + a(v) \cdot \nabla_x f_1 = \sigma(\rho) L(f_2)$

Deterministic equation, diffusive limit, $F = 1$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v)\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon^2}\sigma(\bar{f}^\varepsilon) L f^\varepsilon, \quad L(f) = \bar{f} - f.$$

Hilbert expansion (formal): $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$

$\rightsquigarrow$ order -2 : $L(f_0) = 0$ and $f_0 = \int_V f_0 d\mu = \rho$.

$\rightsquigarrow$ order -1 : $a(v)\cdot\nabla_x\rho = L(f_1) \longrightarrow f_1 = -\sigma(\rho)^{-1} a(v)\cdot\nabla_x\rho$.

$\rightsquigarrow$ order 0 : $\partial_t\rho + a(v)\cdot\nabla_x f_1 = \sigma(\rho) L(f_2)$

$$\longrightarrow \quad \partial_t\rho - \int_V a(v)\cdot\nabla_x(\sigma(\rho)^{-1} a(v)\cdot\nabla_x\rho) d\mu = 0$$

Deterministic equation, diffusive limit, $F = 1$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v)\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon^2}\sigma(\bar{f}^\varepsilon) L f^\varepsilon, \quad L(f) = \bar{f} - f.$$

Hilbert expansion (formal): $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$

$\rightsquigarrow$ order -2 : $L(f_0) = 0$ and $f_0 = \int_V f_0 d\mu = \rho$.

$\rightsquigarrow$ order -1 : $a(v)\cdot\nabla_x\rho = L(f_1) \longrightarrow f_1 = -\sigma(\rho)^{-1} a(v)\cdot\nabla_x\rho$.

$\rightsquigarrow$ order 0 : $\partial_t\rho + a(v)\cdot\nabla_x f_1 = \sigma(\rho) L(f_2)$

$$\longrightarrow \quad \partial_t\rho - \int_V a(v)\cdot\nabla_x(\sigma(\rho)^{-1} a(v)\cdot\nabla_x\rho) d\mu = 0$$

$$\longrightarrow \quad \partial_t\rho - \mathrm{div}\left(\sigma(\rho)^{-1} K\nabla_x\rho\right) = 0,$$

with

$$K := \int_V a(v)\otimes a(v) d\mu(v).$$

Deterministic equation, diffusive limit

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}^\varepsilon) L f^\varepsilon, \ L(f) = \bar{f} F - f.$$

When $\varepsilon \to 0$, the density $\rho^\varepsilon := \int_V f^\varepsilon d\mu$ converges to the solution ρ of the diffusion equation

$$\partial_t \rho - \mathrm{div}(\sigma(\rho)^{-1} K \nabla \rho) = 0$$

with initial data $\rho_0 = \int_V f_0 d\mu$. We assume $\int_V a(v) F(v) d\mu(v) = 0$ and:

$$\forall \varepsilon > 0, \forall (\xi, \alpha) \in S^{N-1} \times \mathbb{R}, \ \ \mu \left(\{ v \in V, |a(v) \cdot \xi + \alpha| < \varepsilon \} \right) \leq \varepsilon^\theta,$$

for some $\theta > 0$.

The stochastic case

We first consider a similar model with time white noise:

$$df^\varepsilon \ + \ \frac{1}{\varepsilon}a(v)\cdot\nabla_x f^\varepsilon \ dt \ = \ \frac{1}{\varepsilon^2}\sigma(\bar{f}^\varepsilon)Lf^\varepsilon dt \ + f^\varepsilon \circ QdW_t,$$

$$x\in\mathbb{T}^N,\ v\in V,\ Lf=\bar{f}F-f.$$

- The noise represents randoms creations/absorptions of photons.
- We expect to obtain a stochastic quasilinear parabolic equation at the limit.
- We adapt the Hilbert expansion method.
- We first have to prove existence of f^ε, we need non degeneracy of a:

$$\forall\varepsilon>0,\,\forall(\xi,\alpha)\in S^{N-1}\times\mathbb{R},\ \ \mu\left(\{v\in V,|a(v)\cdot\xi+\alpha|<\varepsilon\}\right)\leq\varepsilon^\theta,$$

for some $\theta>0$.

Hilbert expansion, stochastic case

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}^\varepsilon) L f^\varepsilon + f^\varepsilon \circ Q dW_t, \quad L(f) = \bar{f} - f.$$

Hilbert expansion (formal): $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$

$\rightsquigarrow$ order -2 : $L f_0 = 0$ and $f_0 = \int_V f_0 d\mu = \rho$.

$\rightsquigarrow$ order -1 : $a(v) \cdot \nabla_x \rho = \sigma(\rho) L f_1 \longrightarrow f_1 = -\sigma(\rho)^{-1} a(v) \cdot \nabla_x \rho$.

$\rightsquigarrow$ order 0 : $\partial_t \rho + a(v) \cdot \nabla_x f_1 = \sigma(\rho) L f_2 + \rho \circ Q dW_t$

Hilbert expansion, stochastic case

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v)\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon^2}\sigma(\bar{f}^\varepsilon) L f^\varepsilon + f^\varepsilon \circ Q dW_t, \quad L(f) = \bar{f} - f.$$

Hilbert expansion (formal): $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$

$\rightsquigarrow$ order -2 : $Lf_0 = 0$ and $f_0 = \int_V f_0 d\mu = \rho$.

$\rightsquigarrow$ order -1 : $a(v)\cdot\nabla_x\rho = \sigma(\rho) L f_1 \longrightarrow f_1 = -\sigma(\rho)^{-1} a(v)\cdot\nabla_x\rho$.

$\rightsquigarrow$ order 0 : $\partial_t\rho + a(v)\cdot\nabla_x f_1 = \sigma(\rho) L f_2 + \rho\circ Q dW_t$

$$\longrightarrow \partial_t\rho - \mathrm{div}\left(\sigma(\rho)^{-1}\Big(\int_V a(v)\otimes a(v) d\mu\Big)\nabla_x\rho\right) = \rho\circ Q dW_t.$$

and

$$\mathrm{div}\left(\sigma(\rho)^{-1}\left(a(v)\otimes a(v) - \int_V a(v)\otimes a(v) d\mu\right)\nabla_x\rho\right) = \sigma(\rho) L f_2.$$

Hilbert expansion, rigorous proof

- We take the solution of the SPDE:

$$\partial_t \rho - \mathrm{div}\left(\sigma(\rho)^{-1}\left(\int_V a(v)\otimes a(v)d\mu\right)\nabla_x\rho\right) = \rho \circ QdW_t.$$

  It is smooth is space provided the noise and initial data are also smooth. (D., De Moor, Hofmanova).
- Define: $f_1 = -\sigma(\rho)^{-1}a(v)\cdot\nabla_x\rho$ and

$$f_2 = -\mathrm{div}\left(\sigma(\rho)^{-1}\left(a(v)\otimes a(v) - \int_V a(v)\otimes a(v)d\mu\right)\nabla_x\rho\right).$$

- Set

$$r^\varepsilon = f^\varepsilon - \rho - \varepsilon f_1 - \varepsilon^2 f_2$$

Hilbert expansion, rigorous proof

- We take the solution of the SPDE:

$$\partial_t \rho - \operatorname{div}\left(\sigma(\rho)^{-1}\left(\int_V a(v)\otimes a(v)d\mu\right)\nabla_x\rho\right) = \rho \circ QdW_t.$$

  It is smooth is space provided the noise and initial data are also smooth. (D., De Moor, Hofmanova).

- Define: $f_1 = -\sigma(\rho)^{-1}a(v)\cdot\nabla_x\rho$ and

$$f_2 = -\operatorname{div}\left(\sigma(\rho)^{-1}\left(a(v)\otimes a(v) - \int_V a(v)\otimes a(v)d\mu\right)\nabla_x\rho\right).$$

- Set

$$r^\varepsilon = f^\varepsilon - \rho - \varepsilon f_1 - \varepsilon^2 f_2$$

  then, with $df_1 = f_{1,d}dt + \Psi_1^\flat dW$,

$$\begin{aligned} dr^\varepsilon + \frac{1}{\varepsilon}a(v)\cdot\nabla_x r^\varepsilon dt &= \frac{1}{\varepsilon^2}\left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]dt \\ &- \varepsilon a(v)\cdot\nabla_x f_2 dt + (f^\varepsilon - \rho - \varepsilon f_1)\, QdW_t \\ &+ G(f^\varepsilon - \rho)\,dt - \varepsilon f_{1,d}dt - \varepsilon\Psi_1^\flat dW_t - \varepsilon^2 df_2. \end{aligned}$$

Hilbert expansion, rigorous proof

$$\begin{aligned} dr^\varepsilon + \frac{1}{\varepsilon} a(v)\cdot\nabla_x r^\varepsilon dt &= \frac{1}{\varepsilon^2}\left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right] dt \\ &\quad - \varepsilon a(v)\cdot\nabla_x f_2 dt + \left(f^\varepsilon - \rho - \varepsilon f_1\right) Q dW_t \\ &\quad + G(f^\varepsilon - \rho)\, dt - \varepsilon f_{1,d} dt \\ &\quad - \varepsilon \Psi_1^\flat dW_t - \ \varepsilon^2 df_2. \end{aligned}$$

The terms: $\frac{1}{\varepsilon} a(v)\cdot\nabla_x r^\varepsilon$ and $\frac{1}{\varepsilon^2}\left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]$ behave well in L^1:

$$\frac{1}{\varepsilon}\int_{\mathbb{T}^N\times V} \left(a(v)\cdot\nabla_x r^\varepsilon\right) sign(r^\varepsilon) d\mu dx = 0,$$

$$\frac{1}{\varepsilon^2}\int_{\mathbb{T}^N\times V} \left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right] sign(r^\varepsilon) d\mu dx \leq 0.$$

Hilbert expansion, rigorous proof

$$
\begin{aligned}
dr^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x r^\varepsilon dt &= \frac{1}{\varepsilon^2} \left[\sigma(\bar{f}^\varepsilon) L(f^\varepsilon) - \sigma(\rho) L(f^\varepsilon - r^\varepsilon) \right] dt \\
&- \varepsilon a(v) \cdot \nabla_x f_2 dt + (f^\varepsilon - \rho - \varepsilon f_1)\, Q dW_t \\
&+ G(f^\varepsilon - \rho)\, dt - \varepsilon f_{1,d} dt \\
&- \varepsilon \Psi_1^\flat dW_t - \varepsilon^2 df_2.
\end{aligned}
$$

The terms: $\frac{1}{\varepsilon} a(v) \cdot \nabla_x r^\varepsilon$ and $\frac{1}{\varepsilon^2} \left[\sigma(\bar{f}^\varepsilon) L(f^\varepsilon) - \sigma(\rho) L(f^\varepsilon - r^\varepsilon) \right]$ behave well in L^1:

$$
\frac{1}{\varepsilon} \int_{\mathbb{T}^N \times V} \left(a(v) \cdot \nabla_x r^\varepsilon \right) sign(r^\varepsilon) d\mu dx = 0,
$$

$$
\frac{1}{\varepsilon^2} \int_{\mathbb{T}^N \times V} \left[\sigma(\bar{f}^\varepsilon) L(f^\varepsilon) - \sigma(\rho) L(f^\varepsilon - r^\varepsilon) \right] sign(r^\varepsilon) d\mu dx \leq 0.
$$

Problem: we cannot use Itô formula for $\|r^\varepsilon\|_{L^1}$.

Hilbert expansion, rigorous proof

$$
\begin{aligned}
dr^\varepsilon + \frac{1}{\varepsilon} a(v)\cdot\nabla_x r^\varepsilon dt \;&= \frac{1}{\varepsilon^2}\left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right] dt \\
&- \varepsilon a(v)\cdot\nabla_x f_2 dt + \left(f^\varepsilon - \rho - \varepsilon f_1\right) Q dW_t \\
&+ G(f^\varepsilon - \rho)\, dt - \varepsilon f_{1,d} dt \\
&- \varepsilon \Psi_1^\flat dW_t - \; \varepsilon^2 df_2.
\end{aligned}
$$

The terms: $\frac{1}{\varepsilon} a(v)\cdot\nabla_x r^\varepsilon$ and $\frac{1}{\varepsilon^2}\left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]$ behave well in L^1. Problem: we cannot use Itô formula for $\|r^\varepsilon\|_{L^1}$.

- We use Itô formula for a δ smoothed version of the L^1 norm.
- This introduces singular terms in the Itô correction: the second derivative of this smoothed L^1 norm is of order $\frac{1}{\delta}$ multiplied by ε^2.
- The use of a modified L^1 norm introduces a term of order $\frac{\delta}{\varepsilon^2}$.

Hilbert expansion, rigorous proof

$$\begin{aligned} dr^\varepsilon + \frac{1}{\varepsilon} a(v)\cdot\nabla_x r^\varepsilon dt &= \frac{1}{\varepsilon^2}\left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right] dt \\ &- \varepsilon a(v)\cdot\nabla_x f_2 dt + (f^\varepsilon - \rho - \varepsilon f_1)\, Q dW_t \\ &+ G(f^\varepsilon - \rho)\, dt - \varepsilon f_{1,d} dt \\ &- \varepsilon \Psi_1^\flat dW_t - \ \varepsilon^2 df_2. \end{aligned}$$

The terms: $\frac{1}{\varepsilon} a(v)\cdot\nabla_x r^\varepsilon$ and $\frac{1}{\varepsilon^2}\left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]$ behave well in L^1. Problem: we cannot use Itô formula for $\|r^\varepsilon\|_{L^1}$.

- We use Itô formula for a δ smoothed version of the L^1 norm.
- This introduces singular terms in the Itô correction: the second derivative of this smoothed L^1 norm is of order $\frac{1}{\delta}$ multiplied by ε^2.
- The use of a modified L^1 norm introduces a term of order $\frac{\delta}{\varepsilon^2}$.

→ We need to kill the noise term of order ε.

→ We need a third corrector f_3 such that

$$\varepsilon^2 df_3 - \sigma(\rho)L(f_3)dt = \Psi_1^\flat dW_t$$

The convergence result

Theorem Let f^ε denote the solution of the kinetic problem

$$df^\varepsilon + \frac{1}{\varepsilon}a(v)\cdot\nabla_x f^\varepsilon \, dt = \frac{1}{\varepsilon^2}\sigma(\bar{f}^\varepsilon)(\bar{f}^\varepsilon F - f^\varepsilon))dt + f^\varepsilon \circ Q dW_t,$$
$$x \in \mathbb{T}^N, \ v \in V.$$

and ρ the solution of the non-linear stochastic partial differential equation

$$\partial_t \rho - \operatorname{div}\left(\sigma(\rho)^{-1} K \nabla_x \rho\right) = \rho \circ Q dW_t,$$

where K denotes the matrix $\left(\int_V a(v)\otimes a(v) d\mu\right)$. Then, the solution f^ε converges as ε tends to 0 to the fluid limit ρ and we have the estimate:

$$\sup_{t\in[0,T]} \mathbb{E}\|f_t^\varepsilon - \rho_t\|_{L^1_{x,v}} \leq C\varepsilon.$$

Another model with "real noise"

We now start with a noise with non vanishing correlation length:

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}^\varepsilon) L f^\varepsilon + \frac{1}{\varepsilon} f^\varepsilon m(\frac{t}{\varepsilon^2}),$$

where $m(t)$ is an ergodic centered markov process with values in a space of functions of x.

- We assume (V, μ) is a measured space, μ is a probability measure, $a \in L^\infty(V; \mathbb{R}^N)$, $N \geq 1$ and $x \in \mathbb{T}^N$.
- The equation is set in $\mathbb{R}_t^+ \times \mathbb{T}_x^N \times V_v$, with initial data $f^\varepsilon(0) = f_0$.
- As before, $L = \bar{f} F - f$ and the velocities are centered: $\int_V a(v) d\mu(v) = \int_V a(v) F(v) d\mu(v) = 0$ and non degenerate:

  $$\forall \varepsilon > 0, \forall (\xi, \alpha) \in S^{N-1} \times \mathbb{R}, \ \ \mu(\{v \in V, |a(v) \cdot \xi + \alpha| < \varepsilon\}) \leq \varepsilon^\theta.$$
- Existence and uniqueness of f^ε is classical under these assumptions.

Diffusion approximation :

We consider a differential equation in $\mathbb{R}^d$ with random coefficients:

$$\frac{dx_t^\varepsilon}{dt} = F(x_t^\varepsilon, m_t^\varepsilon) + \frac{1}{\varepsilon} G(x_t^\varepsilon, m_t^\varepsilon).$$

The driving process m_t^ε scales like $m_t^\varepsilon = m(\varepsilon^{-2} t)$ where m_t is a $\mathbb{R}^d$ valued homogeneous stationary and mixing Markov process. If $G \equiv 0$, then $x_t^\varepsilon \to \overline{x}_t$ where

$$\frac{d\overline{x}}{dt} = \overline{F}(\overline{x}_t), \quad \overline{F}(x) := \int_{\mathbb{R}} F(x,n) d\nu(n),$$

and ν is the invariant measure of m_t.

Diffusion approximation :

We consider a differential equation in $\mathbb{R}^d$ with random coefficients:

$$\frac{dx_t^\varepsilon}{dt} = F(x_t^\varepsilon, m_t^\varepsilon) + \frac{1}{\varepsilon} G(x_t^\varepsilon, m_t^\varepsilon).$$

The driving process m_t^ε scales like $m_t^\varepsilon = m(\varepsilon^{-2} t)$ where m_t is a $\mathbb{R}^d$ valued homogeneous stationary and mixing Markov process. If $G \equiv 0$, then $x_t^\varepsilon \to \overline{x}_t$ where

$$\frac{d\overline{x}}{dt} = \overline{F}(\overline{x}_t), \quad \overline{F}(x) := \int_{\mathbb{R}} F(x,n) d\nu(n),$$

and ν is the invariant measure of m_t. We are interested in the case:

$$G \not\equiv 0, \quad \int_{\mathbb{R}} G(\cdot, n) d\nu(n) \equiv 0 \ ?$$

We concentrate on the case: $G(x,m) = G(x)m$.

Donsker Theorem

Let (ξ_i) be i.i.d centered random variables, with variance $\sigma^2 < +\infty$. Let

$$X_n(t) = \frac{1}{\sigma\sqrt{n}} \left(\xi_1 + \cdots + \xi_{nt}\right), \quad t \in [0,1],$$

the random variable on $C = C([0,1];\mathbb{R})$ defined by linear interpolation between the points $t = i/n$. Then

$$X_n \to \beta$$

where β is a brownian motion on C. The convergence is in law.

Donsker Theorem

Let (ξ_i) be i.i.d centered random variables, with variance $\sigma^2 < +\infty$. Let

$$X_n(t) = \frac{1}{\sigma\sqrt{n}}\left(\xi_1 + \cdots + \xi_{nt}\right), \quad t \in [0,1],$$

the random variable on $C = C([0,1];\mathbb{R})$ defined by linear interpolation between the points $t = i/n$. Then

$$X_n \to \beta$$

where β is a brownian motion on C. The convergence is in law.

$$M_t^\varepsilon = \frac{1}{\varepsilon}\int_0^t m(\frac{s}{\varepsilon^2})ds \sim \varepsilon \sum_0^{[\frac{t}{\varepsilon^2}]} \int_k^{k+1} m(s)ds \to W_t,$$

where $W = (\beta_1, \dots, \beta_d)$ is d dimensional brownian motion.

The perturbed test function method.

Problem : We assume that the driving process m_t^ε scales like $m_t^\varepsilon = m(\varepsilon^{-2}t)$ where m_t is homogeneous and stationary Markov process. We assume that it is mixing with invariant measure ν. Let

$$\frac{d}{dt}x_t^\varepsilon = F(x_t^\varepsilon) + \frac{1}{\varepsilon}G(x_t^\varepsilon)m_t^\varepsilon, .$$

We expect that at the limit $\varepsilon \to 0$, x^ε converges in law to the solution of:

$$dx_t = F(x_t) + G(x_t) \circ dW_t.$$

The perturbed test function method.

Problem : We assume that the driving process m_t^ε scales like $m_t^\varepsilon = m(\varepsilon^{-2}t)$ where m_t is homogeneous and stationary Markov process. We assume that it is mixing with invariant measure ν. Let

$$\frac{d}{dt}x_t^\varepsilon = F(x_t^\varepsilon) + \frac{1}{\varepsilon}G(x_t^\varepsilon)m_t^\varepsilon, .$$

We expect that at the limit $\varepsilon \to 0$, x^ε converges in law to the solution of:

$$dx_t = F(x_t) + G(x_t) \circ dW_t.$$

To prove this we use the generator of $(x^\varepsilon, m^\varepsilon)$. We denote by M the generator of m, then $(x_t^\varepsilon, m_t^\varepsilon)$ has the following generator:

$$\mathcal{L}^\varepsilon \Phi(x,n) = \left(F(x) + \frac{1}{\varepsilon}G(x)n, D_x\Phi(x,n) \right) + \frac{1}{\varepsilon^2}M\Phi(x,n),$$
$$\Phi \in C_b^2(\mathbb{R}^{2d}).$$

Let $v^\varepsilon(t,x,n) = \mathbb{E}(\varphi(x_t^\varepsilon(x), m_t^\varepsilon(n)))$, then $\frac{d}{dt}v^\varepsilon = \mathcal{L}^\varepsilon v^\varepsilon$

The perturbed test function method.

Evolution of $\mathbb{E}(\varphi(x_t^\varepsilon))$:

$$\mathcal{L}^\varepsilon \varphi(x^\varepsilon) = \left(F(x^\varepsilon) + \frac{1}{\varepsilon} G(x^\varepsilon, n), D_x \varphi(x) \right)$$

$\rightsquigarrow$ No information as $\varepsilon \to 0$.

The perturbed test function method.

Evolution of $\mathbb{E}(\varphi(x_t^\varepsilon))$:

$$\mathcal{L}^\varepsilon \varphi(x^\varepsilon) = \left(F(x^\varepsilon) + \frac{1}{\varepsilon} G(x^\varepsilon, n), D_x \varphi(x) \right)$$

$\rightsquigarrow$ No information as $\varepsilon \to 0$.
$\rightsquigarrow$ We try to find correctors $\varphi_1, \varphi_2 \in C_b^2(\mathbb{R}^d \times \mathbb{R}^d)$ such that the perturbed test function

$$\varphi^\varepsilon := \varphi + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2,$$

satisfies

$$\mathcal{L}^\varepsilon \varphi^\varepsilon(x, n) = \mathcal{L}\varphi(x) + \mathcal{O}(\varepsilon)$$

(Papanicolaou, Stroock, Varadhan 77. See the recent book by Fouque, Garnier, Papanicolaou and Solna)

The perturbed test function method.

$$\mathcal{L}^{\varepsilon}\varphi^{\varepsilon}(x,n) = \mathcal{L}\varphi(x) + \mathcal{O}(\varepsilon), \quad \varphi^{\varepsilon} := \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2.$$

Write:

$$\begin{aligned}
&\mathbb{E}(\varphi^{\varepsilon}(x_t^{\varepsilon}, m_t^{\varepsilon})) \\
&= \mathbb{E}(\varphi^{\varepsilon}(x_s^{\varepsilon}, m_s^{\varepsilon})) + \mathbb{E}\left(\int_s^t \mathcal{L}^{\varepsilon}\varphi^{\varepsilon}(x_\sigma^{\varepsilon}, m_\sigma^{\varepsilon})d\sigma\right)
\end{aligned}$$

The perturbed test function method.

$$\mathcal{L}^\varepsilon \varphi^\varepsilon(x,n) = \mathcal{L}\varphi(x) + \mathcal{O}(\varepsilon), \quad \varphi^\varepsilon := \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2.$$

Write:

$$\begin{aligned}&\mathbb{E}(\varphi^\varepsilon(x_t^\varepsilon, m_t^\varepsilon))\\&= \mathbb{E}(\varphi^\varepsilon(x_s^\varepsilon, m_s^\varepsilon)) + \mathbb{E}\left(\int_s^t \mathcal{L}^\varepsilon \varphi^\varepsilon(x_\sigma^\varepsilon, m_\sigma^\varepsilon)d\sigma\right)\end{aligned}$$

$$\varepsilon \to 0 \quad \rightsquigarrow \quad \mathbb{E}(\varphi(x_t)) = \mathbb{E}(\varphi(x_s)) + \mathbb{E}\left(\int_s^t \mathcal{L}\varphi(x_\sigma)d\sigma\right)$$

The perturbed test function method.

$$\mathcal{L}^\varepsilon \varphi^\varepsilon(x,n) = \mathcal{L}\varphi(x) + \mathcal{O}(\varepsilon), \quad \varphi^\varepsilon := \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2.$$

Write:

$$\begin{aligned} &\mathbb{E}(\varphi^\varepsilon(x_t^\varepsilon, m_t^\varepsilon)) \\ &= \mathbb{E}(\varphi^\varepsilon(x_s^\varepsilon, m_s^\varepsilon)) + \mathbb{E}\left(\int_s^t \mathcal{L}^\varepsilon\varphi^\varepsilon(x_\sigma^\varepsilon, m_\sigma^\varepsilon)d\sigma\right) \end{aligned}$$

$$\varepsilon \to 0 \quad \rightsquigarrow \quad \mathbb{E}(\varphi(x_t)) = \mathbb{E}(\varphi(x_s)) + \mathbb{E}\left(\int_s^t \mathcal{L}\varphi(x_\sigma)d\sigma\right)$$

$\rightsquigarrow$ $\mathcal{L}$ is the generator of the limit process.

Equations for the correctors

$$\mathcal{L}^\varepsilon \varphi(x,n) = (F(x) + \frac{1}{\varepsilon} G(x,n), D_x\varphi(x,n)) + \frac{1}{\varepsilon^2} M\varphi(x,n)$$

$$= \mathcal{L}\varphi(x) + \mathcal{O}(\varepsilon), \quad \varphi \in C_b^2(\mathbb{R}^2),$$

$$\varphi^\varepsilon := \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2.$$

We derive

$$M\varphi(x) = 0, \tag{1}$$

$$(G(x)n, D_x\varphi(x)) + M\varphi_1(x,n) = 0, \tag{2}$$

$$(F(x), D_x\varphi) + (G(x)n, D_x\varphi_1(x)) + M\varphi_2(x,n) = \mathcal{L}\varphi(x). \tag{3}$$

The first equation is satisfied since φ does not depend on n.

Equations for the correctors

$$\mathcal{L}^{\varepsilon}\varphi(x,n) = (F(x) + \frac{1}{\varepsilon}G(x,n), D_x\varphi(x,n)) + \frac{1}{\varepsilon^2}M\varphi(x,n)$$

$$= \mathcal{L}\varphi(x) + \mathcal{O}(\varepsilon), \quad \varphi \in C_b^2(\mathbb{R}^2),$$

$$\varphi^{\varepsilon} := \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2.$$

We derive

$$M\varphi(x) = 0, \qquad (1)$$

$$(G(x)n, D_x\varphi(x)) + M\varphi_1(x,n) = 0, \qquad (2)$$

$$(F(x), D_x\varphi) + (G(x)n, D_x\varphi_1(x)) + M\varphi_2(x,n) = \mathcal{L}\varphi(x). \qquad (3)$$

The first equation is satisfied since φ does not depend on n. To solve the second equation, we need to solve the Poisson equation associated to M.

The Poisson equation

We assume that for a large class of functions ψ such that $\int \psi(n) d\nu(n) = 0$ (ν is the invariant law of m_t), the equation

$$M\theta = \psi, \quad \psi \in C_b(\mathbb{R})$$

has a solution in $\theta \in C_b(E)$, unique under the condition $\int \theta(n) d\nu(n) = 0$.

The Poisson equation

We assume that for a large class of functions ψ such that $\int \psi(n) d\nu(n) = 0$ (ν is the invariant law of m_t), the equation

$$M\theta = \psi, \quad \psi \in C_b(\mathbb{R})$$

has a solution in $\theta \in C_b(E)$, unique under the condition $\int \theta(n) d\nu(n) = 0$. It is given by:

$$\theta(n) = M^{-1}\psi(n) := -\int_0^\infty e^{Mt}\psi(n)dt = \int_0^\infty \mathbb{E}\psi(m_t | m(0) = n)dt.$$

e^{Mt} is the transition semi-group associated to m_t.

Equations for the correctors

$$(G(x)n, D_x\varphi(x)) + M\varphi_1(x, n) = 0, \qquad (4)$$

$$(F(x), D_x\varphi) + (G(x)n, D_x\varphi_1(x)) + M\varphi_2(x, n) = \mathcal{L}\varphi(x). \qquad (5)$$

Equations for the correctors

$$(G(x)n, D_x\varphi(x)) + M\varphi_1(x,n) = 0, \tag{4}$$

$$(F(x), D_x\varphi) + (G(x)n, D_x\varphi_1(x)) + M\varphi_2(x,n) = \mathcal{L}\varphi(x). \tag{5}$$

We have assumed $\int_E G(x)nd\nu(n) = 0 \rightsquigarrow$ we obtain

$$\varphi_1 = -M^{-1}(G(x)n, D_x\varphi)$$

Equations for the correctors

$$(G(x)n, D_x\varphi(x)) + M\varphi_1(x,n) = 0, \qquad (4)$$
$$(F(x), D_x\varphi) + (G(x)n, D_x\varphi_1(x)) + M\varphi_2(x,n) = \mathcal{L}\varphi(x). \qquad (5)$$

We have assumed $\int_E G(x)nd\nu(n) = 0 \rightsquigarrow$ we obtain

$$\varphi_1 = -M^{-1}(G(x)n, D_x\varphi)$$

$$\rightsquigarrow \mathcal{L}\varphi(x) = (F(x), D_x\varphi) - \int_E (G(x)n, D_x(M^{-1}G(x)n, D_x\varphi(x)))d\nu(n).$$

This is the generator associated to the SDE:

$$dx = f(X)dt + G(X)oC^{1/2}d\beta$$

where β is a d dimensional brownian motion and C is computed from $\int_E n \otimes M^{-1} n \, d\nu(n)$.

If all coefficients are bounded.
We obtain bounds of the form:

$$\mathbb{E}(\sup_{t\in[0,T]} |x^\varepsilon(t)|^2) \leq c$$

independent on ε and:

$$\mathbb{E}\left(|x^\varepsilon(t) - x^\varepsilon(s)|^4\right) \leq |t-s|^2 + \varepsilon.$$

We write:

$$\mathbb{E}(\varphi^\varepsilon(x^\varepsilon(t), m^\varepsilon(t)))$$
$$= \mathbb{E}(\varphi^\varepsilon(x^\varepsilon(t_0), m^\varepsilon(t_0)) + \int_{t_0}^{t} \mathcal{L}^\varepsilon \varphi^\varepsilon(x^\varepsilon(t), m^\varepsilon(t)))$$

$$\rightsquigarrow \mathbb{E}(\varphi(x^\varepsilon(t))) = \mathbb{E}(\varphi(x^\varepsilon(t_0))) + \mathbb{E}\int_{t_0}^{t} \mathcal{L}\varphi(x^\varepsilon(s))ds + O(\varepsilon),$$

Back to the stochastic kinetic equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}^\varepsilon) L f^\varepsilon + \frac{1}{\varepsilon} f^\varepsilon m^\varepsilon(t),$$

- $m^\varepsilon(t)$ is a centered, mixing markov process with values in a space E of functions of x.
- (V, μ) is a measured space and μ is a probability measure.
- $a \in L^\infty(V; \mathbb{R}^d)$
- $d \geq 1$ and $x \in \mathbb{T}^d$ the d dimensional torus.
- L is a dissipative operator.
- We assume $\int_V a(v) d\mu(v) = 0$ and

$$\forall \varepsilon > 0, \forall (\xi, \alpha) \in S^{N-1} \times \mathbb{R}, \;\; \mu\left(\{v \in V, |a(v) \cdot \xi + \alpha| < \varepsilon\}\right) \leq \varepsilon^\theta,$$

for some $\theta > 0$.

Back to the stochastic kinetic equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}^\varepsilon) L f^\varepsilon + \frac{1}{\varepsilon} f^\varepsilon m^\varepsilon(t),$$

We denote by M the generator of m, then the generator of $(f^\varepsilon, m^\varepsilon)$ is:

$$\mathcal{L}^\varepsilon \varphi^\varepsilon(f, n) \quad = -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(\sigma(\bar{f}) Lf, D\varphi) + \frac{1}{\varepsilon}(nf, D\varphi) + \frac{1}{\varepsilon^2} M\varphi$$

Back to the stochastic kinetic equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}^\varepsilon) L f^\varepsilon + \frac{1}{\varepsilon} f^\varepsilon m^\varepsilon(t),$$

We denote by M the generator of m, then the generator of $(f^\varepsilon, m^\varepsilon)$ is:

$$\begin{aligned} \mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(\sigma(\bar{f}) Lf, D\varphi) + \frac{1}{\varepsilon}(nf, D\varphi) + \frac{1}{\varepsilon^2} M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi \end{aligned}$$

where $Af = a(v) \cdot \nabla_x f$, D is the gradient with respect to f and

$$\mathcal{L}_1 \varphi = -(Af, D\varphi) + (nf, D\varphi)$$

and

$$\mathcal{L}_2 \varphi = (\sigma(\bar{f}) Lf, D\varphi) + M\varphi.$$

Perturbed test function method:

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(\sigma(\bar{f})Lf, D\varphi) + \frac{1}{\varepsilon}(nf, D\varphi) + \frac{1}{\varepsilon^2}M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi) + (nf, D\varphi), \quad \mathcal{L}_2\varphi = (\sigma(\bar{f})Lf, D\varphi) + M\varphi.$$

Perturbed test function method:

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(\sigma(\bar{f})Lf, D\varphi) + \frac{1}{\varepsilon}(nf, D\varphi) + \frac{1}{\varepsilon^2}M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi) + (nf, D\varphi), \quad \mathcal{L}_2\varphi = (\sigma(\bar{f})Lf, D\varphi) + M\varphi.$$

We expect a limit model which is a SPDE with unknown $\rho = \int_V f d\mu(v)$

$\rightsquigarrow$ We use test functions of the form $\varphi(f) = \varphi(\rho)$. And consider $\mathcal{L}^\varepsilon\varphi^\varepsilon$, $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$.

Perturbed test function method:

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(\sigma(\bar{f})Lf, D\varphi) + \frac{1}{\varepsilon}(nf, D\varphi) + \frac{1}{\varepsilon^2}M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi) + (nf, D\varphi), \quad \mathcal{L}_2\varphi = (\sigma(\bar{f})Lf, D\varphi) + M\varphi.$$

We expect a limit model which is a SPDE with unknown $\rho = \int_V f d\mu(v)$

$\rightsquigarrow$ We use test functions of the form $\varphi(f) = \varphi(\rho)$. And consider $\mathcal{L}^\varepsilon \varphi^\varepsilon$, $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$.

$\rightsquigarrow$ Order -2 : $\mathcal{L}_2\varphi = 0$ $\rightarrow$ automatically satisfied.

Perturbed test function method:

$$\begin{aligned} \mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(\sigma(\bar{f})Lf, D\varphi) + \frac{1}{\varepsilon}(nf, D\varphi) + \frac{1}{\varepsilon^2}M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi \end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi) + (nf, D\varphi), \quad \mathcal{L}_2\varphi = (\sigma(\bar{f})Lf, D\varphi) + M\varphi.$$

We expect a limit model which is a SPDE with unknown $\rho = \int_V f d\mu(v)$

$\rightsquigarrow$ We use test functions of the form $\varphi(f) = \varphi(\rho)$. And consider $\mathcal{L}^\varepsilon \varphi^\varepsilon$, $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$.

$\rightsquigarrow$ Order -2 : $\mathcal{L}_2\varphi = 0$ $\rightarrow$ automatically satisfied.

$\rightsquigarrow$ Order -1 : $\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0$

Inversion of $\mathcal{L}_2$:

$$\mathcal{L}_2\psi = (\sigma(\bar{f})Lf, D\psi) + M\psi = \Phi$$

Inversion of $\mathcal{L}_2$:

$$\mathcal{L}_2\psi = (\sigma(\bar{f})Lf, D\psi) + M\psi = \Phi$$

- This is the generator of the process $(g(t; f, n), m(t; f, n))$:

$$\frac{d}{dt}g = \sigma(\bar{g})Lg = \sigma(\bar{g})\left(\int_V g\, d\mu(v) - g\right) = \sigma(\bar{g})\left(\rho - g\right), \; g(0) = f,$$

where m is the driving process starting form n at $t = 0$.

Inversion of $\mathcal{L}_2$:

$$\mathcal{L}_2\psi = (\sigma(\bar{f})Lf, D\psi) + M\psi = \Phi$$

- This is the generator of the process $(g(t;f,n), m(t;f,n))$:

$$\frac{d}{dt}g = \sigma(\bar{g})Lg = \sigma(\bar{g})\left(\int_V g\, d\mu(v) - g\right) = \sigma(\bar{g})\,(\rho - g)\,,\ g(0) = f,$$

where m is the driving process starting form n at $t = 0$.

- Explicit solution : $\rho = \int_V g(t)d\mu(v) = \int_V fd\mu(v)$

$\rightsquigarrow$ $g(t) = e^{-\sigma(\bar{g})t}f + (1 - e^{-\sigma(\bar{g})t})\rho$.

$\rightsquigarrow$

$$\psi(f,n) = \mathcal{L}_2^{-1}\Phi(f,n) = -\int_0^\infty \mathbb{E}\left(\Phi(g(t;f,n);m(t;f,n))\right)dt$$

$$\text{if } \int_E \Phi(\rho, n)d\nu(n) = 0.$$

Perturbed test function method:

$$\begin{aligned}\mathcal{L}^{\varepsilon}\varphi^{\varepsilon}(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(\sigma(\bar{f})Lf, D\varphi) + \frac{1}{\varepsilon}(nf, D\varphi) + \frac{1}{\varepsilon^2}M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi) + (nf, D\varphi), \quad \mathcal{L}_2\varphi = (\sigma(\bar{f})Lf, D\varphi) + M\varphi$$

$\rightsquigarrow$ Order -2 : $\mathcal{L}_2\varphi = 0 \to$ automatically satisfied

$\rightsquigarrow$ Order -1 : $\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0$

$$\int_E \mathcal{L}_1\varphi(\rho, n)d\nu(n) = \int_E -(A\rho, D\varphi) + (n\rho, D\varphi)d\nu(n) = 0$$

Perturbed test function method:

$$\mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) = -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(\sigma(\bar{f})Lf, D\varphi) + \frac{1}{\varepsilon}(nf, D\varphi) + \frac{1}{\varepsilon^2}M\varphi$$
$$= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi) + (nf, D\varphi), \quad \mathcal{L}_2\varphi = (\sigma(\bar{f})Lf, D\varphi) + M\varphi$$

$\rightsquigarrow$ Order -2 : $\mathcal{L}_2\varphi = 0 \rightarrow$ automatically satisfied

$\rightsquigarrow$ Order -1 : $\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0$

$$\int_E \mathcal{L}_1\varphi(\rho,n)d\nu(n) = \int_E -(A\rho, D\varphi) + (n\rho, D\varphi)d\nu(n) = 0$$

$$\rightarrow \varphi_1 = -\mathcal{L}_2^{-1}\mathcal{L}_1\varphi = \int_0^\infty -(Ag(t;f,n), D\varphi) + (m(t;f,n)g(t,f,n), D\varphi)dt$$

Perturbed test function method:

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(Lf, D\varphi) + \frac{1}{\varepsilon}(m, D\varphi) + \frac{1}{\varepsilon^2} M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi) + (m, D\varphi), \quad \mathcal{L}_2\varphi = (Lf, D\varphi) + M\varphi$$

⇝ Order -2 : $\mathcal{L}_2\varphi = 0$ → automatically satisfied

⇝ Order -1 : $\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0$

$$\int_E \mathcal{L}_1\varphi(\rho, n) d\nu(n) = \int_E -(A\rho, D\varphi) + (n\rho, D\varphi) d\nu(n) = 0$$

$$\to \varphi_1 = -\mathcal{L}_2^{-1}\mathcal{L}_1\varphi = -\left(A\left(\sigma(\rho)^{-1} f\right), D\varphi\right) - (fM^{-1}n, D\varphi).$$

Perturbed test function method:

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(Lf, D\varphi) + \frac{1}{\varepsilon}(m, D\varphi) + \frac{1}{\varepsilon^2} M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi) + (m, D\varphi), \quad \mathcal{L}_2\varphi = (Lf, D\varphi) + M\varphi$$

⇝ Order -2 : $\mathcal{L}_2\varphi = 0 \rightarrow$ automatically satisfied

⇝ Order -1 :
$\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0, \quad \varphi_1 = -(A\left(\sigma(\rho)^{-1} f\right), D\varphi) - (fM^{-1}n, D\varphi).$

⇝ Order 0 : $\mathcal{L}_1\varphi_1 + \mathcal{L}_2\varphi_2 = \mathcal{L}\varphi \rightarrow \mathcal{L}\varphi(\rho) = \int_E \mathcal{L}_1\varphi_1(\rho, n) d\nu(n).$

Limit generator:

$$\mathcal{L}\varphi = \int_E \mathcal{L}_1\varphi_1 d\nu(n)$$
$$= (\mathcal{A}\rho, D\varphi) - \int_E \Big((\rho n M^{-1} n, D\varphi(\rho)) + D^2\varphi(\rho)\cdot(\rho M^{-1} n, \rho n) \Big) d\nu(n).$$

where

$$\mathcal{A}\rho = \mathrm{div}\big(\big(\sigma(\rho)^{-1}\big)\, K\nabla\rho\big)$$

This is the generator of

$$d\rho \quad = \mathrm{div}(K\nabla\rho)dt + \rho\circ Q^{1/2}dW(t)$$
$$= \mathrm{div}(K\nabla\rho)dt + \frac{1}{2}F\rho + \rho Q^{1/2}dW(t).$$

Limit $\varepsilon \to 0$

- To complete the proof, we need to prove tightness of the laws of $\rho^\varepsilon = \overline{f}^\varepsilon$.
- Bound in $L^2(\mathbb{T}^N)$: Take $\varphi(f) = \|f\|_{L^2}^2$
  (weigthed norm: $\|f\|_{L^2}^2 = \int_{\mathbb{T}^N \times V} f^2(x,v) F^{-1}(v) dx dv$.)
- It is not a function of $\rho = \overline{f}$ but it is possible to compute correctors and obtain a bound on $\|f^\varepsilon\|_{L^2(\mathbb{T}^N)}$ in $L^\infty(0,T)$.
- This implies tightness in $C([0,T]; H^{-\eta}(\mathbb{T}^N))$, $\eta > 0$.
- This is not sufficient to deal with the nonlinear term.

Limit $\varepsilon \to 0$

- We also have a bound $\frac{1}{\varepsilon}\|Lf^\varepsilon\|_{L^2}$ in $L^2(0,T)$.
- $\varepsilon\partial_t f^\varepsilon + a(v)\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon}\sigma(\bar{f}^\varepsilon)Lf^\varepsilon + m^\varepsilon f^\varepsilon$ is bounded in $L^2(0,T;L^2(\mathbb{T}^N))$.
- Averaging Lemma: *Assume*

$$\forall\varepsilon>0,\ \forall(\xi,\alpha)\in S^{N-1}\times\mathbb{R},\ \ \mu\left(\{v\in V, |a(v)\cdot\xi+\alpha|<\varepsilon\}\right)\leq\varepsilon^\theta,$$

for some $\theta > 0$.
If f^ε *and* $\varepsilon\partial_t f^\varepsilon + a(v)\cdot\nabla_x f^\varepsilon$ *are bounded in* $L^2(0,T;L^2(\mathbb{T}^N))$
then $\rho^\varepsilon = \bar{f}^\varepsilon$ *is bounded in* $L^2(0,T;H^s)$, *for* $s<\theta/2$.

- We get tightness in $L^2(0,T;L^2(\mathbb{T}^N))$

Limit $\varepsilon \to 0$

Theorem Let $f_0^\varepsilon \in L^2_{x,v}$ and

$$\rho_0 := \int_V f_0 d\mu.$$

Under the above assumptions on the velocities a and on the driving process m^ε, we have: for all $\eta > 0$, the density $\rho^\varepsilon := \int_V f^\varepsilon d\mu$ converge in law in $C([0,T]; H^{-\eta})$ and in $L^2(0,T;H^s)$ to the solution ρ of the equation

$$\begin{aligned} d\rho &= \mathrm{div}(\sigma(\rho)^{-1}K\nabla\rho)dt + \rho \circ Q^{1/2}dW(t), \text{ in } \mathbb{R}_t^+ \times \mathbb{T}^d, \\ &= \mathrm{div}(\sigma(\rho)^{-1}K\nabla\rho)dt + \frac{1}{2}F\rho + \rho Q^{1/2}dW(t), \text{ in } \mathbb{R}_t^+ \times \mathbb{T}^d, \end{aligned}$$

with initial data ρ_0, where W is a cylindrical Wiener process on $L^2(\mathbb{T}^d)$, Q is a nuclear operator on $L^2(\mathbb{T}^d)$ determined by the correlation of m.

Coefficient Q in the limit model

It is associated to a kernel k:

$$Qf(x) = \int_{\mathbb{T}^d} k(x,y)f(y)dy, \quad f \in L^2(\mathbb{T}^d),$$

where

$$k(x,y) := \mathbb{E}\int_{\mathbb{R}} m(0)(y)m(t)(x)dt, \quad x,y \in \mathbb{T}^d.$$

The Itô correction :

$$F(x) = k(x,x).$$

The noise as a force (linear case):

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon + \frac{1}{\varepsilon^2} m^\varepsilon \cdot \nabla_v f = \frac{1}{\varepsilon^2} L f^\varepsilon.$$

- $m^\varepsilon(t)$ is a centered mixing markov process with values in a space of functions E.
- $v \in V = \mathbb{T}^d$.
- $a(v) = v$.
- $Lf = \rho F - f$ where F is an equilibrium function satisfying:

$$\int_V vF(v) d\mu(v) = 0.$$

The noise as a force (linear case):

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon + \frac{1}{\varepsilon^2} m^\varepsilon \cdot \nabla_v f = \frac{1}{\varepsilon^2} L f^\varepsilon.$$

We denote by M the generator of m, then the generator of $f^\varepsilon, m^\varepsilon$ is given by:

$$\begin{aligned} \mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(Lf, D\varphi) - \frac{1}{\varepsilon^2}(mBf, D\varphi) + \frac{1}{\varepsilon^2} M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi \end{aligned}$$

where $Af = v \cdot \nabla_x f$, $Bf = \nabla_v f$, D is the gradient with respect to f and now

$$\mathcal{L}_1\varphi = -(Af, D\varphi)$$

and

$$\mathcal{L}_2\varphi = (Lf, D\varphi) - (mBf, D\varphi) + M\varphi$$

Perturbed test function method

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(Lf, D\varphi) - \frac{1}{\varepsilon}(mBf, D\varphi) + \frac{1}{\varepsilon^2}M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi), \quad \mathcal{L}_2\varphi = (Lf, D\varphi) - (mBf, D\varphi) + M\varphi$$

We expect a limit model which is a SPDE with unknown $\rho = \int_V f d\mu(v)$

$\rightsquigarrow$ We use test functions of the form $\varphi(f) = \varphi(\rho)$.

$\rightsquigarrow$ Order -2 : $\mathcal{L}_2\varphi = 0$ $\rightarrow$ automatically satisfied

Perturbed test function method

$$\begin{aligned}\mathcal{L}^{\varepsilon}\varphi^{\varepsilon}(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(Lf, D\varphi) - \frac{1}{\varepsilon}(mBf, D\varphi) + \frac{1}{\varepsilon^2}M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi), \quad \mathcal{L}_2\varphi = (Lf, D\varphi) - (mBf, D\varphi) + M\varphi$$

We expect a limit model which is a SPDE with unknown $\rho = \int_V f d\mu(v)$

$\leadsto$ We use test functions of the form $\varphi(f) = \varphi(\rho)$.

$\leadsto$ Order -2 : $\mathcal{L}_2\varphi = 0$ $\rightarrow$ automatically satisfied

$\leadsto$ Order -1 : $\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0$

Inversion of $\mathcal{L}_2$:

$$\mathcal{L}_2\varphi = (Lf, D\varphi) + (mBf, D\varphi) + M\varphi$$

Inversion of $\mathcal{L}_2$:

$$\mathcal{L}_2\varphi = (Lf, D\varphi) + (mBf, D\varphi) + M\varphi$$

This is the generator of the process $(g(t; f, n), m(t; f, n))$:

$$\frac{d}{dt}g = Lg - mBg = \rho F - g - m \cdot \nabla_v g, \quad g(0) = f,$$

where m is the driving process starting form n at $t = 0$.

Inversion of $\mathcal{L}_2$:

$$\mathcal{L}_2\varphi = (Lf, D\varphi) + (mBf, D\varphi) + M\varphi$$

This is the generator of the process $(g(t; f, n), m(t; f, n))$:

$$\frac{d}{dt}g = Lg - mBg = \rho F - g - m \cdot \nabla_v g, \quad g(0) = f,$$

where m is the driving process starting form n at $t = 0$.

Explicit solution: $\rho = \int_V g(t) d\mu(v) = \int_V f d\mu(v)$

$\rightsquigarrow$

$$g(t, x, v) = e^{-t} f(x, v - M_t) + \int_0^t e^{-(t-s)} \rho(x) F(v + M_s - M_t) ds.$$

where $M_t = \int_0^t m(s, x, n) ds$.

Inversion of $\mathcal{L}_2$:

$$\mathcal{L}_2\varphi = (Lf, D\varphi) + (mBf, D\varphi) + M\varphi$$

This is the generator of the process $(g(t; f, n), m(t; f, n))$:

$$\frac{d}{dt}g = Lg - mBg = \rho F - g - m \cdot \nabla_v g, \quad g(0) = f,$$

where m is the driving process starting form n at $t = 0$.

Explicit solution: $\rho = \int_V g(t)d\mu(v) = \int_V fd\mu(v)$

$\rightsquigarrow$

$$g(t, x, v) = e^{-t}f(x, v - M_t) + \int_0^t e^{-(t-s)}\rho(x)F(v + M_s - M_t)ds.$$

where $M_t = \int_0^t m(s, x, n)ds$.

$\rightsquigarrow$ $\mathcal{L}_2^{-1}\psi(f, n) = -\int_0^\infty \mathbb{E}\left(\psi(g(t; f, n); m(t; f, n))\right) dt$ if

$$\int_E \psi\Big(\int_{-\infty}^0 e^s \rho F(v - \int_s^0 m(\sigma, n)d\sigma)ds, n\Big)d\nu(n) = 0.$$

Perturbed test function method:

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(Lf, D\varphi) + \frac{1}{\varepsilon}(m, D\varphi) + \frac{1}{\varepsilon^2} M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi), \quad \mathcal{L}_2\varphi = (Lf, D\varphi) - (mBf, D\varphi) + M\varphi$$

$\rightsquigarrow$ Order -2 : $\mathcal{L}_2\varphi = 0$ $\rightarrow$ automatically satisfied

Perturbed test function method:

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(Lf, D\varphi) + \frac{1}{\varepsilon}(m, D\varphi) + \frac{1}{\varepsilon^2}M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi), \quad \mathcal{L}_2\varphi = (Lf, D\varphi) - (mBf, D\varphi) + M\varphi$$

$\rightsquigarrow$ Order -2 : $\mathcal{L}_2\varphi = 0 \rightarrow$ automatically satisfied

$\rightsquigarrow$ Order -1 : $\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0$. It is possible to invert $\mathcal{L}_2$

Perturbed test function method:

$$\mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) = -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(Lf, D\varphi) + \frac{1}{\varepsilon}(m, D\varphi) + \frac{1}{\varepsilon^2} M\varphi$$
$$= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi), \quad \mathcal{L}_2\varphi = (Lf, D\varphi) - (mBf, D\varphi) + M\varphi$$

$\rightsquigarrow$ Order -2 : $\mathcal{L}_2\varphi = 0$ $\rightarrow$ automatically satisfied

$\rightsquigarrow$ Order -1 : $\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0$. It is possible to invert $\mathcal{L}_2$:

$$\varphi_1 = -(Af, D\varphi) - (\mathrm{div}(fM^{-1}n), D\varphi(f)).$$

Perturbed test function method:

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(Lf, D\varphi) + \frac{1}{\varepsilon}(m, D\varphi) + \frac{1}{\varepsilon^2} M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi), \quad \mathcal{L}_2\varphi = (Lf, D\varphi) - (mBf, D\varphi) + M\varphi$$

$\rightsquigarrow$ Order -2 : $\mathcal{L}_2\varphi = 0$ $\to$ automatically satisfied

$\rightsquigarrow$ Order -1 : $\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0$. It is possible to invert $\mathcal{L}_2$:

$$\varphi_1 = -(Af, D\varphi) - (\mathrm{div}(fM^{-1}n), D\varphi(f)).$$

$\rightsquigarrow$ Order 0 : $\mathcal{L}_1\varphi_1 + \mathcal{L}_2\varphi_2 = \mathcal{L}\varphi$

$$\to \mathcal{L}\varphi = -\int_E \mathcal{L}_1\varphi_1\left(\int_{-\infty}^0 e^s \rho F(v - \int_s^0 m(\sigma,n)d\sigma)ds\right)d\nu(n)$$

.

Limit generator:

Very long computations ...

Limit generator:

Very long computations ...
We obtain the limit SPDE:

$$d\rho = \mathrm{div}((K+H)\nabla\rho)dt + \mathrm{div}(\rho G) + \mathrm{div}(\rho \circ Q^{1/2} dW(t)), \text{ in } \mathbb{R}_t^+ \times \mathbb{T}^d$$

The operator Q:

$$Qf(x) = \int_{\mathbb{T}^d} k(x,y) f(y) dy, \quad f \in L^2(\mathbb{T}^d),$$

where

$$k(x,y) := \mathbb{E} \int_{\mathbb{R}} m(0)(y) \otimes m(t)(x) dt, \quad x, y \in \mathbb{T}^d.$$

The extra (deterministic) diffusion:

$$H(x) := \mathbb{E} \int_0^\infty e^{-s} m(0)(x) \otimes m(t)(x) dt, \quad x \in \mathbb{T}^d.$$