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M. Visan

Symplectic Non-squeezing for the cubic NLS on $\mathbb{R}^2$:

Q: Symplectic Non-squeezing in infinite volume?

Canonical Example of a Symplectic Manifold.

$\mathbb{R}^n \times \mathbb{R}^n \ni z = (x, p)$

$\omega = dp_1 \wedge dx_1 + \cdots + dp_n \wedge dx_n$ the symplectic form.

$H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ Hamiltonian

$$\omega(\cdot, \dot{z}) = dH(\cdot) \iff \begin{cases} \dot{x}_j = \dfrac{dH}{dp_j} \\ \dot{p}_j = \dfrac{-dH}{dq_j} \end{cases}$$

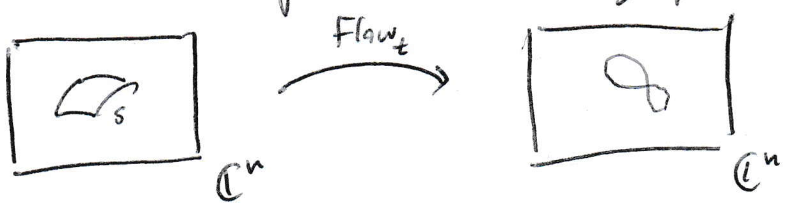
Rephrase: $\mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{C}^n$,

$(x, p) \longmapsto x + ip$

$\omega(z, J) = -\operatorname{Im} \langle z, J \rangle_{\mathbb{C}^n} = -\operatorname{Im} \sum' z_j \bar{J}_j$

Symplectic Hilbert Space: $(\mathcal{H}, \omega)$, $\omega(z, J) = -\operatorname{Im} \langle z, J \rangle_{\mathcal{H}}$.

Hamiltonian flows preserve the symplectic form.



$\text{Flow}_t^*(\omega) = \omega$

$$\int_S \omega = \int_{\text{Flow}_t(S)} \omega$$

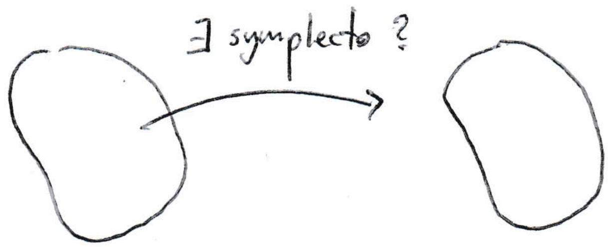
If $n=1$, then ω is a volume form

$\text{Area}(S) = \text{Area}(\text{Flow}_t(S))$

$n \geq 2$, $\underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}$ — " — — " —.

<u>Theorem</u> (Liouville) Hamiltonian flows preserve volume.

<u>Question</u>:

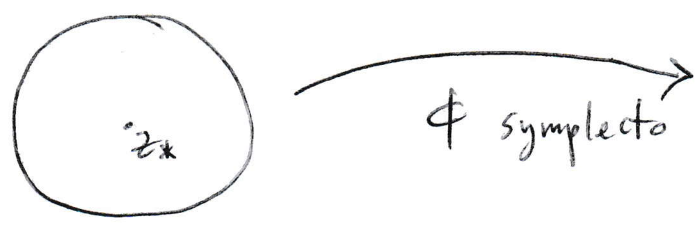


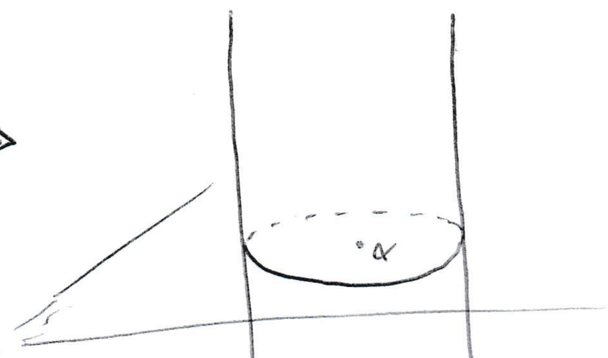
$\text{Vol}(\text{blob}_1) = \text{Vol}(\text{blob}_2)$.

<u>Answer</u>: If $n=1$, Yes!

If $n \geq 2$, No!

<u>Gromov's non-squeezing theorem</u>.





$B(z_*, R) = \{ z \in \mathbb{C}^n : |z - z_*| \leq R \}$

If $\phi(B(z_*, R)) \subseteq C_r(\alpha, \ell)$,

then $r \geq R$.

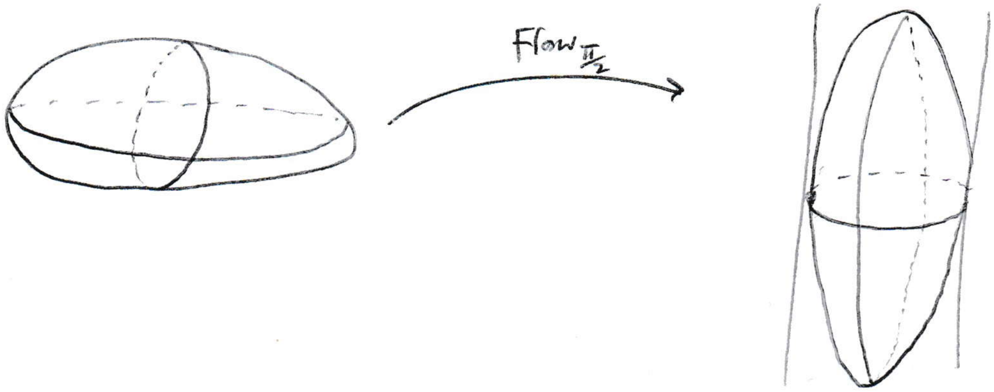
$Cr(\alpha, \ell) = \{ z \in \mathbb{C}^n : |\langle z, \ell \rangle - \alpha| \leq r \}$

$\alpha \in \mathbb{C}$, $\ell \in \mathbb{C}^n$, $\|\ell\|_2 = 1$.

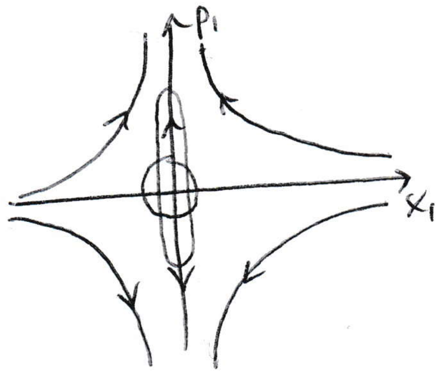
$$C_r = \{ z \in \mathbb{C}^n : |x_1 - \text{Re}\,\alpha|^2 + |p_1 - \text{Im}\,\alpha|^2 \leq r^2 \} = C_r(\alpha, e_1).$$

<u>Squeezing</u> $\quad H = x_1 p_2 - x_2 p_1$

1)



2) $\quad H = -x_1 p_1 - x_2 p_2$



squeezes x_1 and x_2.

3) $\quad H = -x_1 p_1 + x_2 p_2 \quad$ squeezes x_1 and p_2.

<u>Model</u>

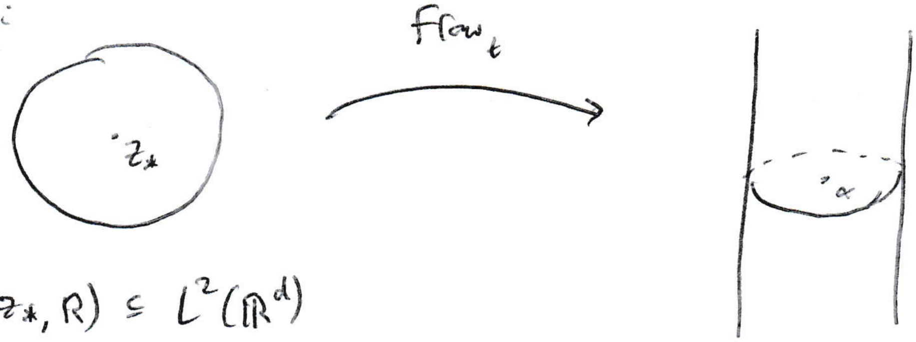
$$(NLS) \quad \begin{cases} (i\partial_t + \Delta) u = |u|^p u, & p > 0 \\ t \in \mathbb{R},\ x \in \mathbb{R}^d \ (x \in \mathbb{T}^d). \end{cases}$$

$$H(u) = \int \frac{1}{2} |\nabla u(t,x)|^2 + \frac{1}{p+2} |u(t,x)|^{p+2}\, dx$$

$$\omega : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \longrightarrow \mathbb{R}, \quad \omega(u,v) = \operatorname{Im} \int u \cdot \bar{v}\, dx$$

$$= -\operatorname{Im} \langle u, v \rangle_{L^2}.$$

Goal:



$B(z_*, R) \subseteq L^2(\mathbb{R}^d)$

$$C_r(\alpha, \ell) = \{ f \in L^2(\mathbb{R}^d) : |\langle f, \ell \rangle - \alpha| \leq r \}$$

$\alpha \in \mathbb{C}$, $\ell \in L^2(\mathbb{R}^d)$, $\|\ell\|_2 = 1$.

If $\mathrm{Flow}_t(B(z_*, R)) \subseteq C_r(\alpha, \ell) \implies r \geq R$.

If $p \leq \frac{4}{d}$, then (NLS) is well-posed on $L^2(\mathbb{R}^d)$.

If $p < \frac{4}{d}$, $T_{lwp} \gtrsim \|u_0\|_2^{-c}$

If $p = \frac{4}{d}$, $T_{lwp}(u_0)$.

Theorem (Dodson): If $u_0 \in L^2(\mathbb{R}^d)$, then $\exists\,!$ global solution u to (NLS) with $p = 4/d$ and

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d}} \, dx\, dt \leq C(\|u_0\|_2).$$

Model
$$\begin{cases} (i\partial_t + \Delta) u = |u|^2 u \\ u(0) = u_0 \in L^2(\mathbb{R}^2) \end{cases}$$

Theorem (Killip, V., Zhang) Let $z_* \in L^2(\mathbb{R}^2)$, $\ell \in L^2(\mathbb{R}^2)$ with $\|\ell\|_2 = 1$, $\alpha \in \mathbb{C}$, $0 < r < R < \infty$, $T > 0$.

Then $\exists\, u_0 \in B(z_*, R)$ s.t.

$$\left| \langle u(T), \ell \rangle - \alpha \right| > r.$$

History

Kuksin $\Phi(t)$ = linear operator + smooth compact perturbation on $\mathbb{T}^d$.

Bourgain cubic NLS on $\mathbb{T}$

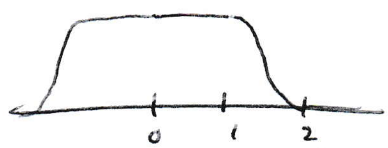
CKSTT KdV on $\mathbb{T}$

Roumegoux BBM on $\mathbb{T}$

Mendelson cubic KG on $\mathbb{T}^3$.

Problem Prove a Dodson-type result for

$$(i\partial_t + \Delta) u = P_{\leq 1}\left(|P_{\leq 1} u|^2 P_{\leq 1} u \right).$$

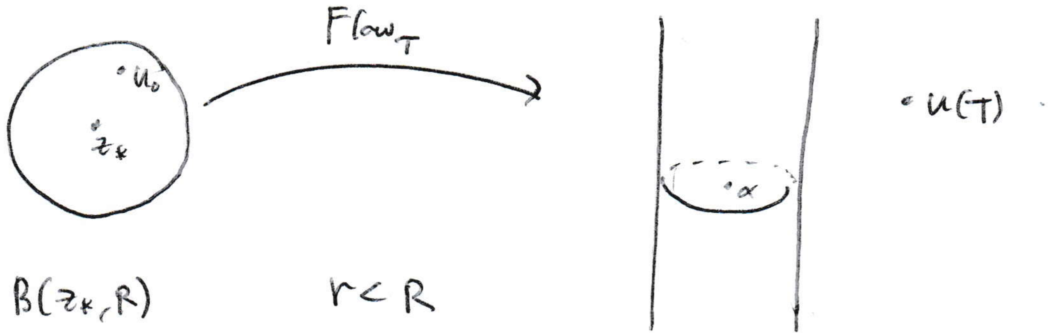


Advantage of the periodic setting.

$$u(t,x) = \sum_{k \in \mathbb{Z}^d}' \hat{u}(t,k)\, e^{ikx}$$

Idea: Fourier truncate to $|k| \leq N$.

Sketch of Proof



$B(z_*, R)$ $\qquad r < R \qquad$ $C_r(\alpha, \ell)$.

Choose $\quad N_n \longrightarrow \infty, \quad L_n \longrightarrow \infty$.

Consider $\quad (i\partial_t + \Delta) u_n = P_{\le N_n}\left(|P_{\le N_n} u|^2 P_{\le N_n} u \right)$.

$$u_n(0) = u_{n,0} \in \mathcal{H}_n := \left\{ f \in L^2\left(\mathbb{R}^2 / L_n \mathbb{Z}^2 \right) : P_{>2N_n} f = 0 \right\}$$

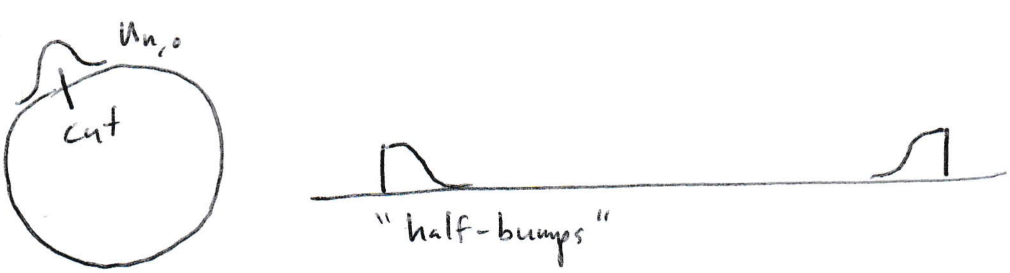
By Gromov

$$\exists\, u_{n,0} \in B_{\mathcal{H}_n}(z_*, R) \text{ s.t. } \left| \langle u_n(T), \ell \rangle - \alpha \right| > \frac{R+r}{2}.$$

Naive idea:

$$\begin{array}{ccc} u_{n,0} & \xmapsto{\ NLS_n\ } & u_n(T) \\ \big\downarrow & & \big\downarrow \\ u_{\infty,0} & \xmapsto{\ ?\,NLS\,?\ } & u_{\infty,T} \end{array}$$

Need to avoid:



soln: take large torus, use "almost" finite speed of propagation and select $L_n \gg N_n$.

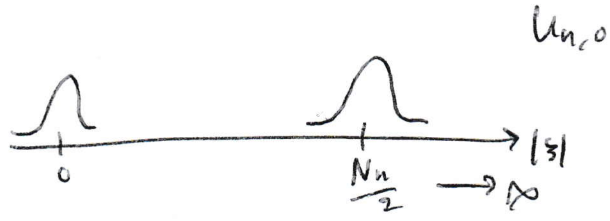
M. Visan. (7)

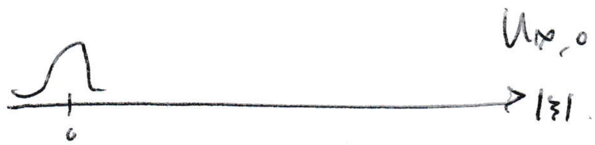
Uniform bounds: set $L_n = \infty$.

Replace $P_{\le N_n}$ w/ $\tilde{P}_{\le N_n}$ with "very slow" tails in (NLS_n), allows for good space-time bounds with $L_n < \infty$.

How to prove weak limits are solutions?

Scenario:





use a nonlinear profile decomposition.