Lagrange Prize Lecture

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Andrew J. Majda

An Applied Math Perspective on Climate Science, Turbulence, and Other Complex Systems

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Lecture Available at Majda's Website:
http://math.nyu.edu/faculty/majda/
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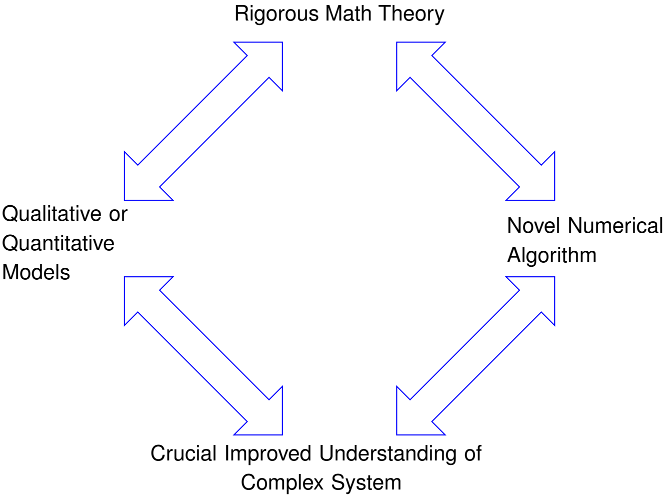
Grand Challenge in Climate Science as Extremely Complex System

- Important societal impacts: predicting long range weather forecasting (intraseasonal to interannual) and short term (decadal) climate change.
- Turbulent dynamical system: huge phase space and large dimension of instabilities.
- Other examples, engineering turbulence, neural science, material science.
- Need statistical, stochastic, thinking combined with nonlinear dynamics ideas.

Central Applied Math/Science Issues

1. Accurate prediction and representation of suitable statistics for observations from nature.
2. Model error: lack of physical understanding and inadequate resolution due to curse of ensemble size, computational overload in generating even small number of ensemble members is overwhelming.
3. Uncertainty quantification (UQ) accurate bounds for 1) and 2).
4. Low order models which achieve 1) and 3) while coping with 2) in an optimal fashion.
5. Rapid data assimilation or filtering to aid prediction.

Modern Applied Math Paradigm



Three Turbulent Dynamical Systems with Crucial Qualitative Features of Aspects of Climate Science and Other Complex System

1. The truncated Burgers-Hopf (TBH) equations (Majda and Timofeyev, *PNAS* 2000)

   Low frequency variability of atmosphere.

2. The Lorenz 96 (L-96) model (Lorenz 1996)

   Shear or baroclinic turbulence (weather) of midlatitudes.

3. The MMT model of dispersive wave turbulence (Majda, McLaughlin, Tabak, *JNLS.* 1997)

   Simple model with coherent structure and wave radiation with direct and inverse turbulent cascades of energy like many geophysical systems.

1. The TBH equations (Majda and Timofeyev)

The finite Galerkin truncation of inviscid Burgers equation

$$(u_\Lambda)_t + \frac{1}{2} P_\Lambda (u_\Lambda^2)_x = 0,$$

$$u_\Lambda = \sum_{|k| \le \Lambda} \hat{u}_k e^{ikx}, \qquad P_\Lambda u = u_\Lambda.$$

Conserved quantities (no others!):

$$\int u_\Lambda \ \text{(momentum)}, \qquad \frac{1}{2}\int u_\Lambda^2 \ \text{(energy)}, \qquad \frac{1}{3}\int u_\Lambda^3 \ \text{(cube)}.$$

Intuition: as a shock steepens, it cannot form and scatters energy back to large scale (low frequency variability).

Statistical predictions:

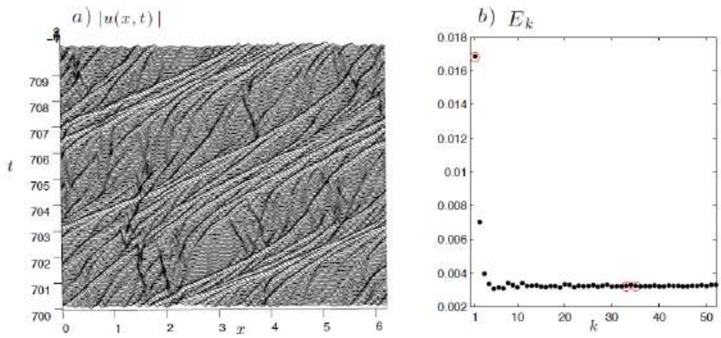
1. Equipartition of energy.
2. Correlation scaling law, large scales decorrelate more slowly, no separation of scales.
3. Confirmed in simulations for Λ, $\Lambda \cong 40$ modes.

Hamiltonian system with $\frac{1}{3}\int u_\Lambda^3 = \mathcal{H}$, the Hamiltonian! (Abramov, Kovacic and Majda, *CPAM* 2003)

The TBHi equations

$$(u_\Lambda)_t + P_\Lambda(\tfrac{1}{2}u_\Lambda^2)_x = H[u_\Lambda],$$

where H is the Hilbert transform.



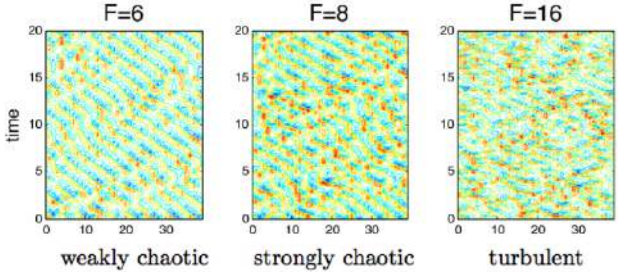
Uses of TBH:

- Prediction (Kleeman, Majda, Timofeyev, *PNAS* 2002).
- Deterministic and stochastic low order modelling (Majda, Timofeyev, Vanden-Eijnden).
- Data assimilation by stochastic superresolution (Branicki and Majda, *JCP* 2012).

2. The Lorenz 96 (L-96) model

$$\frac{du_j}{dt} = (u_{j+1} - u_{j-2})u_{j-1} - u_j + F, \qquad j = 0, \ldots, J-1, \qquad \text{with} \qquad J = 40.$$

Depending on the forcing value F the system will exhibit completely different dynamical features.



	F	λ_1	N^+	KS	T_{corr}
Weakly chaotic	6	1.02	12	5.547	8.23
Strongly chaotic	8	1.74	13	10.94	6.704
Fully turbulent	16	3.945	16	27.94	5.594

Here, λ_1 denotes the largest Lyapunov exponent, N^+ denotes the dimension of the expanding subspace of the attractor, KS denotes the Kolmogorov-Sinai entropy, and T_{corr} denotes the decorrelation time of energy-rescaled time correlation function.

(See Majda and Harlim (M-H) book – "**Filtering Complex Turbulent System**", 2012)

The L-96 model

- Mimics midlatitude baroclinic waves along midlatitude circle.
- Energy of weather moves eastward but individual (Rossby) waves move westwards.
- Uses asymmetric energy conserving nonlinearity with forty modes where $F = 8$ is Lorenz original value.

Uses of L-96 model:

- Predictability (Lorenz and Emanuel).
- Data assimilation (See *MWR, Phys D, M-H* Book).
- Climate change response (Abramov and Majda, Nonlinearity 2007; Majda and Di Qi, *JNLS*. 2015).
- UQ and low order modeling (Sapsis and Majda, *PNAS* 2013) .

3. The MMT equation

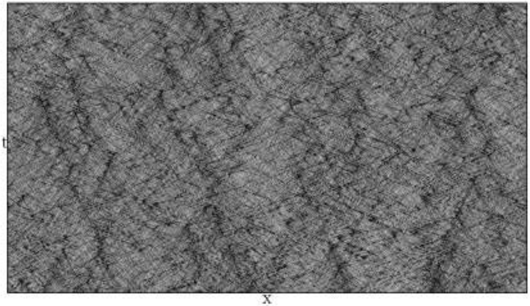
The MMT equation (Majda, McLaughlin and Tabak, 1997; Cai and M.M.T., *Phys. D* 2001)

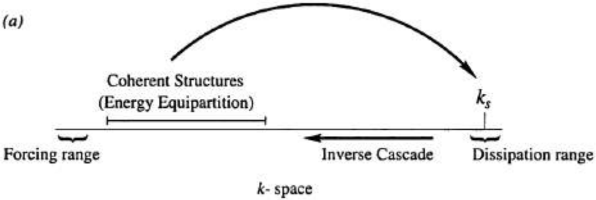
$$iu_t = |\partial_x|^{\frac{1}{2}} u + \lambda |u|^2 u - iAu + F.$$

Here we consider the case with the focusing nonlinearity, $\lambda = -1$, which induces spatially coherent 'solitonic' excitations at random spatial locations.

- The instability of collapsing solitons radiate energy to large scales producing direct and inverse turbulent cascades.
- In geophysical applications energy oftern flows from small scales to large scales (inverse cascade) creating a challenge for reduced modelling.
- Fractional dispersion are crucial with completely different behavior from NLS equation!

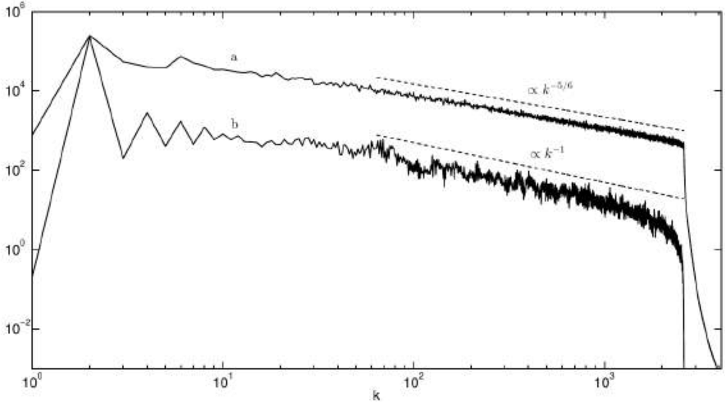
Visualization of $|\psi(x,t)|$ from simulation with $F_0 = 0.0163$; darker colors indicate higher amplitudes. Here the number of Fourier modes are $64^2 \approx 4000$.





From Cai etal, *Physica D* 2001.

High-resolution reference simulations

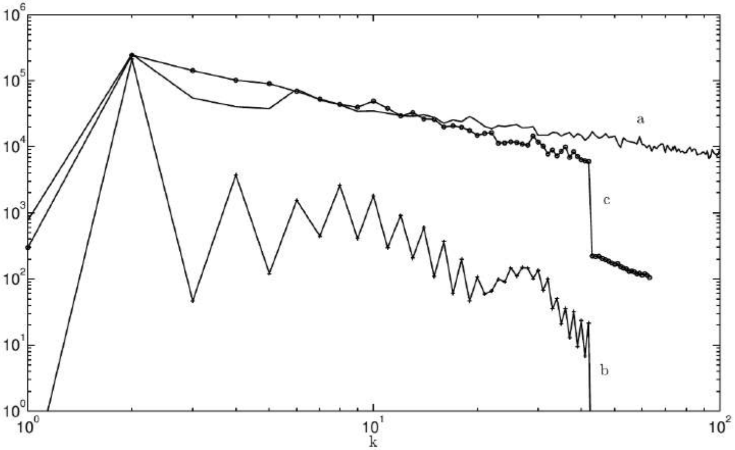


Simulation (a) uses $F_0 = 0.0163$; (b) uses $F_0 = 0.01625$. Both simulations are damped only for $2600 < |k| < 4096$ and $|k| = 1$.

Uses of MMT model:

1. Novel low order modelling: stochastic superparameterization (Majda and Grooms, *JCP* 2013; Grooms and Majda, *Comm. Math. Sci.* 2014).
2. Novel data assimilation (Branicki and Majda, *JCP* 2012; Grooms, Lee and Majda, *JCP* 2014)
3. Extreme event prediction (Cousins and Sapsis, *Phys. D.* 2014)

Stochastic Superparameterization in MMT



Spectra from simulations with $1/64$ as many points as the reference simulation (a), with no eddy terms (b) and with eddy terms (c).

Stochastic Superparameterization

1. A general framework for stochastic subgridscale modelling with no scale separation and no small-scale equilibration based on the Gaussian closure approximation and the point approximation.
2. Success in a difficult test problem with no scale separation ($k^{-5/6}$ spectra), coherent structures, dispersive waves, and an inverse cascade from unresolved scales into the large scales.
3. Overcome *curse of ensemble size* with judicious model error.

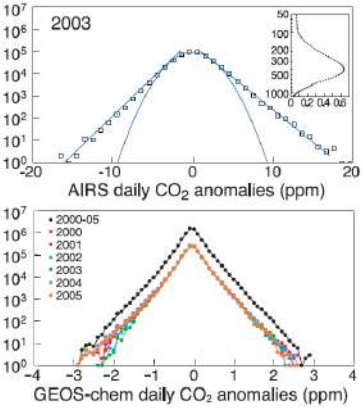
– See research expository article Majda and Grooms, *JCP* 2013; Grooms and Majda, PNAS, *JCP* 2013 for geophysical turbulence.

– See Khouider, Biello and Majda, *Comm. Math. Sci.* 2010; Deng, Khouider and Majda, *JAS* 2015 for stochastic multi-cloud model.

Rigorous Mathematical Models with Intermittency and Extreme Events

Neelin et al, *GRL*, 2011, CO and CO_2, probability distribution function (PDF) exhibit intermittency and extreme event in observations.

– Fat tails (nearly exponential) compared with Gaussian.



Model CO_2 as passive tracer with a mean gradient.

Exactly solvable test models with realistic features in climate change science

$$\frac{\partial T}{\partial t} + \vec{v}(\vec{x}, t) \cdot \nabla T = \kappa \Delta T.$$

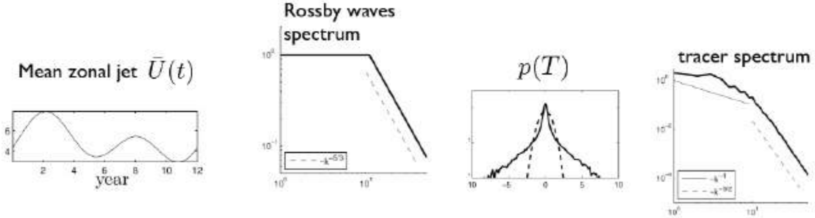
Turbulent velocity

$$\vec{v}(\vec{x}, t) = (U(t), v(x, t))^T$$

$U(t)$, $v(x, t)$ known random field.

Passive tracer with mean gradient

$$T = \alpha y + T'(x, t)$$



(Research expository: Majda and Gershgorin, *Phil. Roy. Soc.* 2013; Bourlioux and Majda, *Phys Fluids* 2002; Majda and Gershogorin, *PNAS* 2011, 2012)

- Model error and stochastic parameterization

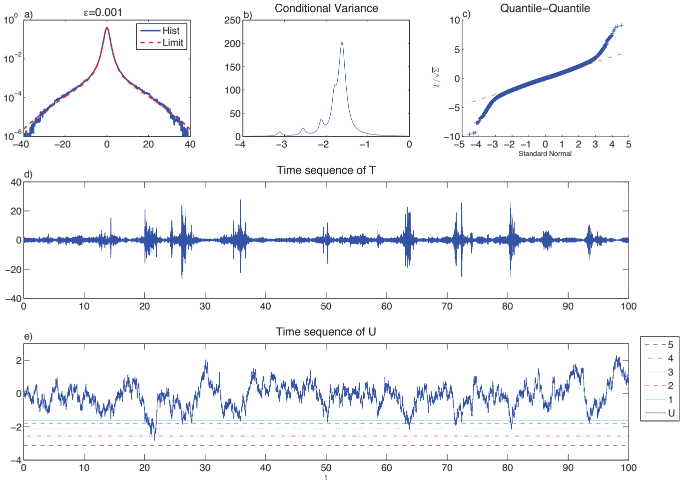
$$\frac{\partial T^M}{\partial t} + \vec{\bar{v}}^M \cdot \nabla T^M = (\kappa + \kappa_{eddy}) \Delta T^M + \sigma_T \dot{W}.$$

- Extreme event prediction with model error (Di Qi and Majda, *Phys. D* 2015)

Rigorous analysis of extreme events

(Majda and Xin Tong, *Nonlinearity* 2015)

Rigorous PDF v.s. Simulation



Intermittent bursts occur when the random mean flow, $U(t)$, gets close to a certain resonant set, rigorous analysis.

Information-Theoretic Framework, Information Barrier and Improving Predictive Skill with Model Error

(Majda and Gershgorin, *PNAS*, 2011, 2012)

Information-theoretic framework is extensively applied in the study of model error, predictive skill and data assimilation. The following three information-theoretic measures are widely used,

1. The Shannon entropy of the residual $S(\mathbf{u} - \mathbf{u}^M)$ measures the uncertainty in the model $\mathbf{u}^M$ compared with the truth $\mathbf{u}$. It is the surrogate for the RMS error in the path-wise sense.

$$S(\mathbf{U}) := -\int p(\mathbf{U}) \ln p(\mathbf{U}) d\mathbf{U}, \qquad \mathbf{U} = \mathbf{u} - \mathbf{u}^M.$$

2. The mutual information $M(\mathbf{u}, \mathbf{u}^M)$ measures the dependence between $\mathbf{u}$ and $\mathbf{u}^M$. It is the surrogate for the anomaly pattern correlation in the path-wise sense.

$$M(\mathbf{u}, \mathbf{u}^M) := \int\int p(\mathbf{u}, \mathbf{u}^M) \ln \frac{p(\mathbf{u}, \mathbf{u}^M)}{\pi(\mathbf{u})\pi^M(\mathbf{u}^M)} d\mathbf{u} d\mathbf{u}^M.$$

3. The relative entropy $P(\pi, \pi^M)$ quantifies the lack of information or model error in the statistics of $\mathbf{u}^M$ relative to that of $\mathbf{u}$. It is also an indicator of assessing the disparity in the amplitudes and peaks between $\mathbf{u}^M$ and $\mathbf{u}$.

$$P(\pi, \pi^M) := \int \pi(\mathbf{u}) \ln \frac{\pi(\mathbf{u})}{\pi^M(\mathbf{u})} d\mathbf{u}.$$

A simple example with an intrinsic barrier for improving model sensitivity

Perfect model:

$$\frac{du}{dt} = au + v + F, \qquad \frac{dv}{dt} = qu + Av + \sigma \dot{W}.$$

Smooth Gaussian measure if $a + A < 0, \quad aA - q > 0.$

Imperfect model:

$$\frac{du_M}{dt} = -\gamma_M u_M + F_M + \sigma_M \dot{W}_M, \qquad \gamma_M > 0.$$

- Climate fidelity for imperfect model

$$\frac{F_M}{\gamma_M} = -\frac{AF}{aA - q}, \quad \frac{\sigma_M^2}{2\gamma_M} = \frac{\sigma^2}{2(a + A)(aA - q)} \equiv E.$$

- Response to change in forcing

$$\delta u = -\frac{A}{aA - q}\delta F, \quad \delta u_M = \frac{1}{\gamma_M}\delta F.$$

- Information model error in response to change in forcing

$$P(\pi_\delta, \pi_\delta^M) = \frac{1}{2}E^{-1}\left|-\frac{A}{aA - q} - \frac{1}{\gamma_M}\right|^2 |\delta F|^2 \qquad \text{for perfect model fidelity.}$$

With $A > 0$, the attempt to minimize the information theoretic model error is futile because no finite minimum over γ_M is achived and necessarily $\gamma_M \to \infty$ in the approach to the minimum – **intrinsic information barrier**.

Improving the predictive skill of imperfect models for complex systems in their response to external forcing

$$\text{Perfect system:} \qquad u_t = F(u) + \sigma(u)\dot{W},$$

$$\text{Perturbed system:} \qquad u_t^\delta = F(u^\delta) + \delta f(t) + \sigma(u^\delta)\dot{W}.$$

- Equilibrium statistical fidelity – a necessary condition.
- Combining the information theory with linear response theory in improving the predictive fidelity.
  - Leading order correction to the statistics of functional $A(u)$ for small δ,

    $$\delta\langle A(u)\rangle = \int_0^t R_A(t-s)\delta f(s)ds,$$

    $R_A(t)$ – the linear response operator calculated through correlation functions in the unperturbed climate.
  - Improving the predictive skill by minimizing the model error to response,

$$P\left(\pi_\delta, \pi_\delta^M\right) = S(\pi_{G,\delta}) - S(\pi_\delta) + \frac{1}{2}\bar{\sigma}^{-2}\left(\int_0^t (R_{\bar{u}}(t-s) - R_{\bar{u}}^M(t-s))\delta f(s)ds\right)^2 + \frac{1}{4}\bar{\sigma}^{-4}\left(\int_0^t (R_{\bar{\sigma}^2}(t-s) - R_{\bar{\sigma}^2}^M(t-s))\delta f(s)ds\right)^2 + O(\delta^3).$$

Examples and applications.

- Improving response in the turbulent tracer model: Majda and Gershgorin, *PNAS* 2011; Di Qi and Majda, 2015
- Low order models and climate change forcing: Majda and Di Qi, *JNLS*, 2015
- Intermittent models: Branicki and Majda, *Nonlinearity* 2012
- Model error in data assimilation: Branicki and Majda, *Comm. Math. Sci.*, 2014
- Low order model prediction: Nan Chen and Majda, *GRL* 2014, *MWR* 2015, *MCWF* 2015

Lessons for UQ and Failure of Polynomial Chaos

– Research expository: Majda and Branicki, *DCDS*, 2012.
– Exactly solvable test models for polynomial chaos: Branicki and Majda, *Comm. Math. Sci.*, 2013.

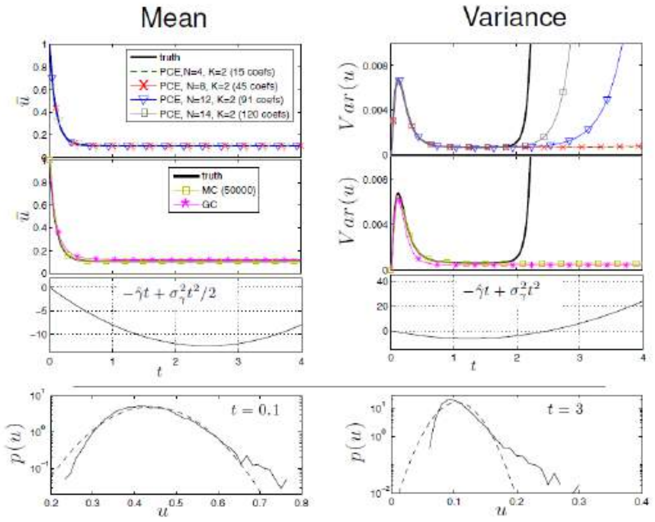
Failure of PC and even Monte Carlo with very large ensemble size.

- Simplest example: Linear ODE with parametric uncertainty

$$\dot{u} = -(\gamma + \sigma_\gamma \xi)u + f(t).$$

where parametric uncertainty is Gaussian random variable $\sigma_\gamma \xi$, ξ is $\mathcal{N}(0, 1)$.
Easy to exactly solve equations for mean, variance and any moment in time.

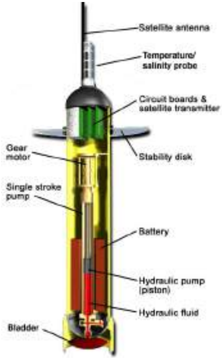
Failure of polynomial chaos and straightforward Monte Carlo.

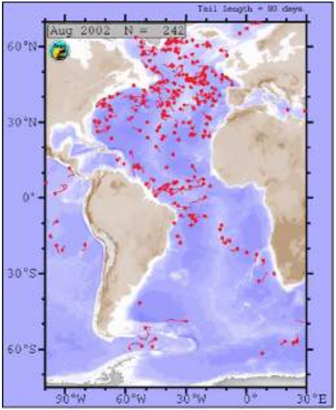


Both PC with 120 coefficients and MC with $50,000$ samples fail to predict the variance with any accuracy!

Inverse Problems and Data Assimilation

Lagrangian Tracers: Oceanography





C. Jones, A. Apte, A. Stuart, ...

Inverse Problem: Noisy Lagrangian Tracers in Filtering Geophysical Flows

First rigorous math theory

(Nan Chen, Majda, Xin Tong, *Nonlinearity* 2014, *JNLS* 2015)

Observing L noisy trajectories $X_j(t)$,

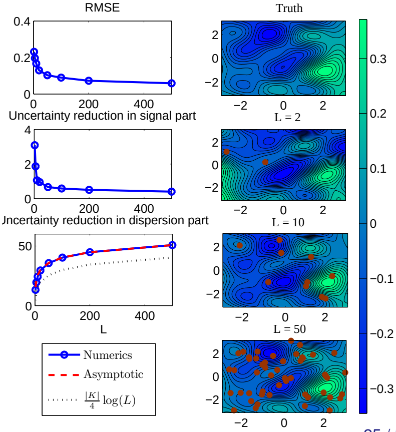
$$\frac{dX_j}{dt} = v(X_j(t), t) + \sigma_j \dot{W}_j.$$

Recover or estimate the velocity $\vec{v}$.

- Inherent nonlinearity in measurement.
- Build exact closed analytic formulas for the optimal filter for the velocity field.
- Prove a mean field limit at long times.

1. Recovering random incompressible flows

- Show an exponential increase in the number of tracers for reducing the uncertainty by a fixed amount – **a practical information barrier**.

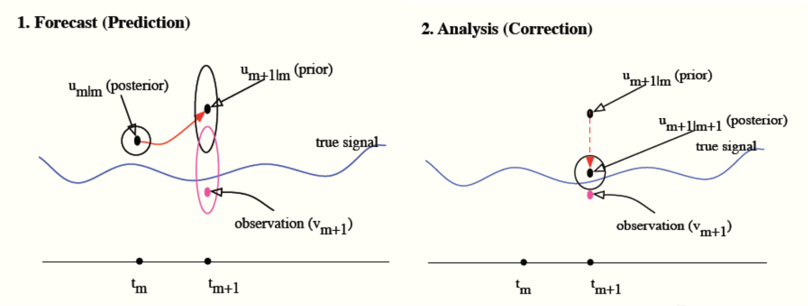


2. Noisy Lagrangian tracers for filtering random rotating compressible flows

(Nan Chen, Majda, Xin Tong, *JNLS* 2015)

- Rotating shallow water models with multiscale features:
  - Slow modes – random incompressible geostrophically balanced (GB) flows.
  - Fast modes – random rotating compressible gravity waves.
- Highly nonlinear observations mixing GB and gravity modes.
- Proposing different filters.
  - Full filter – full forecast model & tracer observations.
  - Ideal reference GB filter – GB forecast model & GB observations.
  - Reduced filter – GB forecast model & mixed observations – a practical inexpensive imperfect filter.
- Rigorous math theory: Comparable high skill in recovering GB modes for all the filters in the geophysical scenario with small Rossby number.

Filtering the Turbulent Signals



Filtering is a two-step process involving statistical prediction of the state variables through a forward operator followed by an analysis step at the next observation time which corrects this prediction on the basis of the statistical input of noisy observations of the system.

Practical Issue

- Turbulent dynamical system.
- Huge phase space, $N = O(10^6, 10^8, etc)$.
- Nonlinearity, small ensemble size $M = O(50, 100)$.

Applied algorithm

- Finite ensemble Kalman filter, (Evensen, 1995; C. Bishop, J. Anderson 2001; Kalnay, 2013). See M-H book.

Applied math

- Stuart, Reich,...

Central issues

- Why does EnKF often work well to estimate the mean with $M \leq N$?

Surprising pathology

- Catastrophic filter divergence. For filtering forced dissipative system with absorbing ball property such as L-96 model, EnKF can explode to machine infinity in finite time! (Harlim and Majda 2008; Gottwald and Majda, *NPG* 2013)

Well posedness of EnKF

- Kelly, Law, Stuart, *Nonlinearity* 2014.

Rigorous nonlinear stability for finite ensemble Kalman filter (EnKF)

(Xin Tong, Majda, Kelly, *Nonlinearity* 2015)

Filter divergence – a potential flaw for EnKF:

- Catastrophic filter divergence: the ensemble members diverging to infinity,
- Lack of stability: the ensemble members being trapped in locations far from the true process.

Finding practical conditions and modifications to rule out filter divergence with rigorous analysis:

- Ruling out catastrophic filter divergence by establishing an energy principle for the filter ensemble.
- Looking for energy principles inherited by the Kalman filtering scheme.
- Looking for modification schemes of EnKF that ensures an energy principle and preserving the original EnKF performance.
- Verifying the nonlinear stability of EnKF through geometric ergodicity.

Multiscale framework:

- Studying filter stability under sparse and incomplete observation networks.
- Providing theoretical guidelines for multiscale data assimilation methods and multiscale forecast models to prevent catastrophic filter divergence.

Rigorous example of catastrophic divergence:

- For filtering a nonlinear map with absorbing ball property (Kelly, Majda, Xin Tong, *PNAS* 2015).

Outstanding problem: Why and when is there accuracy in mean for $M \leq N$?

A Hierarchy of Models for Predicting and Understanding

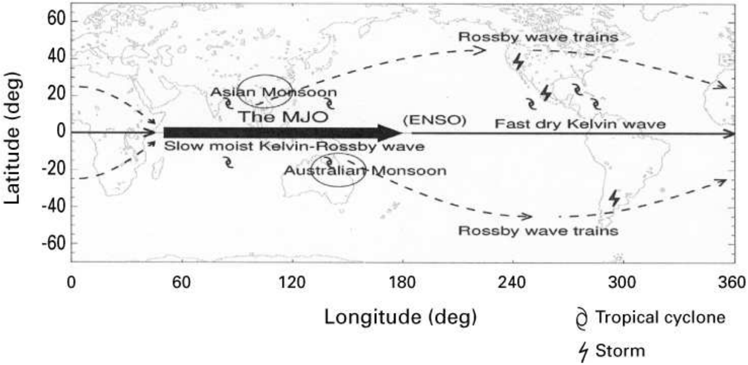
I. The Madden-Julian Oscillation (MJO)

the dominant component of tropical intraseasonal variability

Global impact of MJO

The MJO affects

- El Niño-Southern Oscillation
- Monsoons
- Tropical cyclones
- Midlatitude predicability

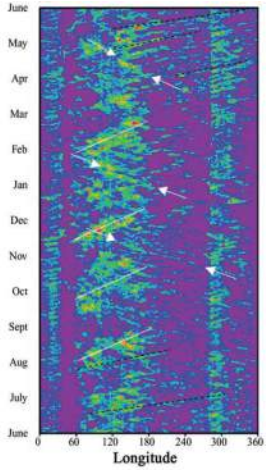


from Moncrieff, Shapiro, Slingo, & Molteni, "Collaborative research at the intersection of weather and climate", *WMO Bulletin*, 2007.

MJO diagnostics in observations and GCMs

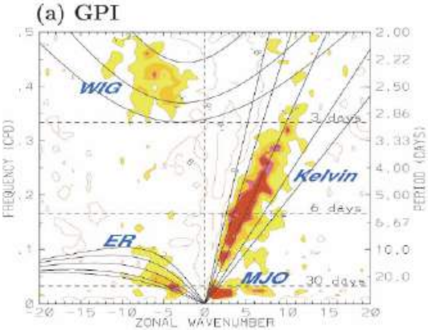
Precipitation

2000-2001 (from Zhang 2005)



Spectral Power

(from Lin et al. 2006)



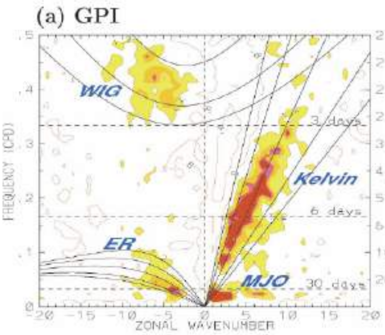
MJO: slow eastward propagation ≈ 5 m/s.

MJO: peculiar dispersion relation $\frac{d\omega}{dk} \approx 0$.

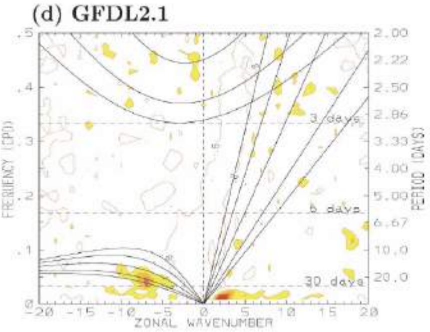
MJO is an envelope of smaller-scale convection/waves

MJO diagnostics in observations and GCMs

Observations



Global Climate Model (GCM)



from Lin etal. (2006)

GCMs typically don't adequately represent
convectively coupled equatorial waves and the MJO.

Novel Nonlinear Time-Series Techniques to Capture both Intermittency & Low-Frequency Variability in Massive Data Sets

Nonlinear Laplacian Spectral Analysis (NLSA)

(Giannakis and Majda, *PNAS* 2012)

NLSA combines:

- Lagged embedding
- Machine learning
- Adaptive weights
- Spectral entropy criteria

NLSA is applied to the data sets of dimensions $O(10^6)$!

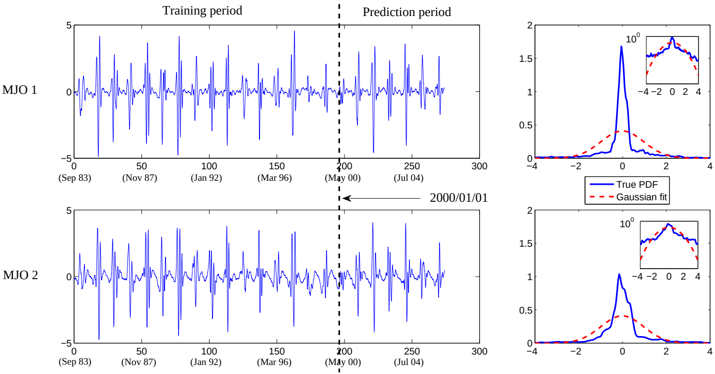
- Applications with W. Tung and E. Szekely to OLR for cloud patterns from tropics, MJO and Monsoon.
- Applications with M. Bushuk to Arctic sea ice reemergence.

Predicting Cloud Patterns of MJO through Low-Order Stochastic Models

(Nan Chen, Majda, Giannakis, *GRL* 2014)

(Nan Chen, Majda, *MWR* 2015)

NLSA Time-Series Techniques $\Longrightarrow$ 2 components of MJO Cloud Patterns



Intermittent bursts of MJO activity

Physics-Constrained Low-Order Nonlinear Stochastic Model for Predicting MJO Cloud Patterns (MJO1, MJO2)

Physics-Constrained Low-Order Stochastic Model

$$
\begin{aligned}
du_1 &= (-d_u\, u_1 + \gamma\, (v + v_f(t))\, u_1 - (a + \omega_u)\, u_2)dt + \sigma_u\, dW_{u_1},\\
du_2 &= (-d_u\, u_2 + \gamma\, (v + v_f(t))\, u_2 + (a + \omega_u)\, u_1)dt + \sigma_u\, dW_{u_2},\\
dv &= (-d_v\, v - \gamma\, (u_1^2 + u_2^2))dt + \sigma_v\, dW_v,\\
d\omega_u &= (-d_\omega \omega_u + \hat{\omega}_u)dt + \sigma_\omega\, dW_\omega,
\end{aligned}
$$

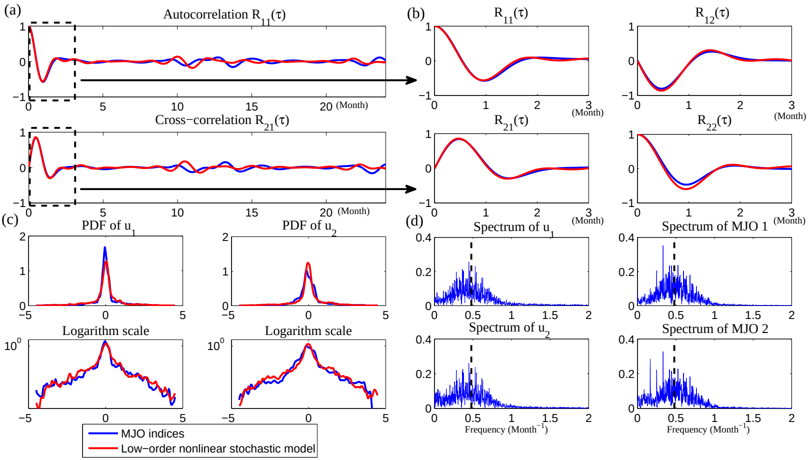
with

$$
v_f(t) = f_0 + f_t \sin(\omega_f t + \phi).
$$

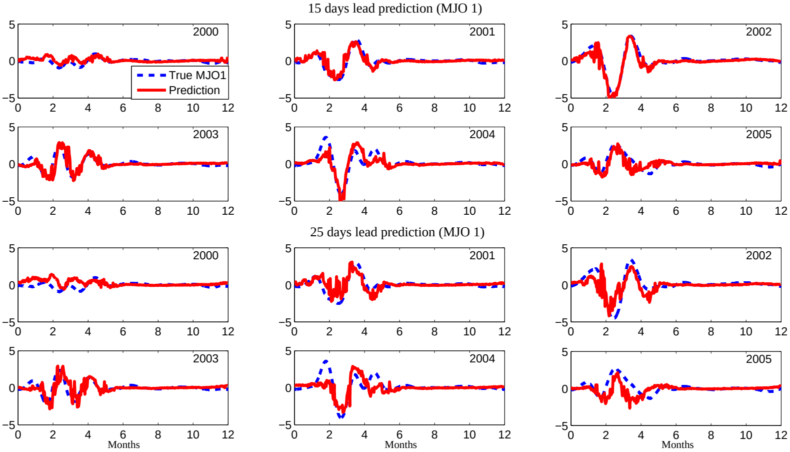
- Observed variables u_1, u_2: MJO 1 and MJO 2 indices from NLSA.
- Hidden variables v, ω: stochastic damping and stochastic phase.
- Energy-conserving nonlinear interactions between (u_1, u_2) and (v, ω_u) (Majda and Harlim, Nonlinearity 2012).
- Effective data assimilation algorithm incorporating into prediction scheme.

Calibration of parameters using *Information Theory* (Robust parameters)

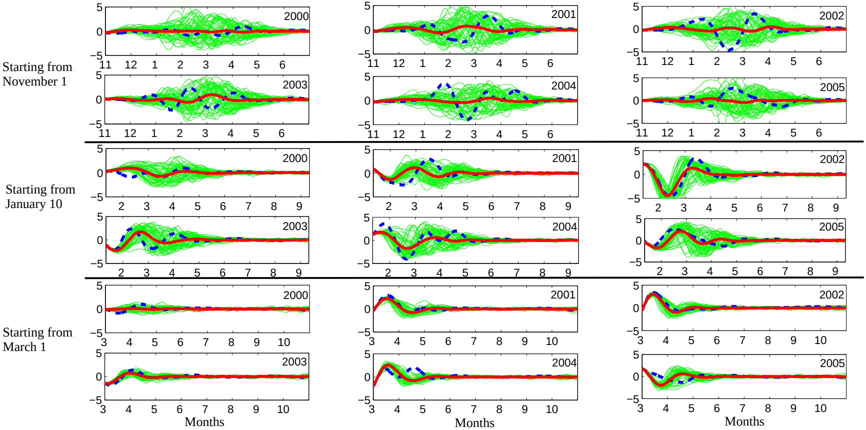
Model vs. Observations: Non-Gaussian statistics match



Skillful prediction at 15- and 25-days lead times



Varying Start Date of Prediction



Ensemble spread $\Longleftrightarrow$ long-range forecast uncertainty is captured

II. Hierarchy of Models for MJO

A New Model for the MJO

Majda and Stechmann 2009 *PNAS*
"The Skeleton of Tropical Intraseasonal Oscillations"

Majda and Stechmann 2009 *JAS*
"Nonlinear Dynamics and Regional Variations in the MJO Skeleton"

Simultaneously captures all three fundamental features of the MJO skeleton:

1. Eastward propagation speed of ≈ 5 m/s
2. Peculiar dispersion relation of $\frac{d\omega}{dk} \approx 0$
3. Horizontal quadrupole vortex structure

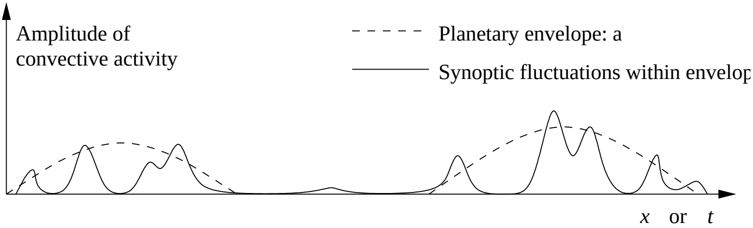
Fundamental mechanism proposed for MJO skeleton

Minimal, nonlinear oscillator model

Neutrally stable interactions between

1. planetary-scale, lower-tropospheric moisture: q
2. sub-planetary-scale, convection/wave activity: a

Based on multi-scale concepts



Tacit assumption: primary instabilities/damping occur on synoptic scales

Minimal nonlinear oscillator model

$$
\boxed{
\begin{aligned}
u_t - yv &= -p_x \\
yu &= -p_y \\
0 &= -p_z + \theta \\
u_x + v_y + w_z &= 0 \\
\theta_t + w &= \bar{H}a - s^\theta \\
q_t - \tilde{Q}w &= -\bar{H}a + s^q \\
\\
a_t &= \Gamma q a
\end{aligned}
}
$$

Linearized primitive equations

- Equatorial long-wave scaling
- Coriolis term: equatorial β-plane approx.

+

Dynamic equation for convective activity

- q: lower tropospheric moisture anomaly
- a: amplitude of convective activity envelope

Key mechanism: positive q creates a tendency to enhance convective activity a

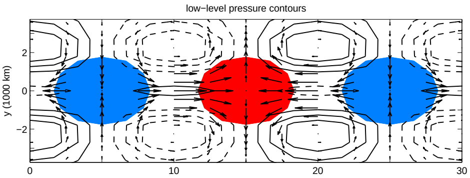
Minimal number of parameters: s^θ, $\tilde{Q}$, Γ

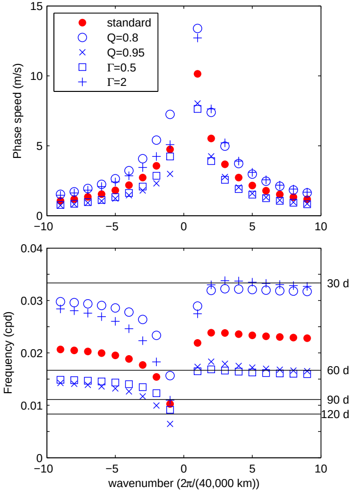
Observational evidence, Waliser 2003

Linear Theory

Simultaneously captures all three fundamental features of the MJO skeleton:

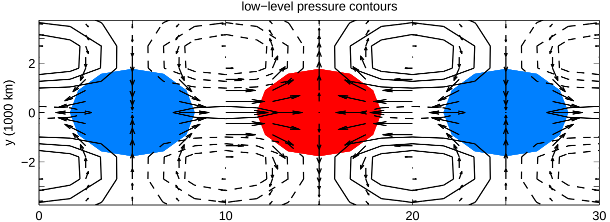
1. Eastward propagation speed of ≈ 5 m/s
2. Peculiar dispersion relation of $\frac{d\omega}{dk} \approx 0$
3. Horizontal quadrupole vortex structure





MJO Skeleton Index for identifying & monitoring MJO activity

Theoretical prediction of MJO structure



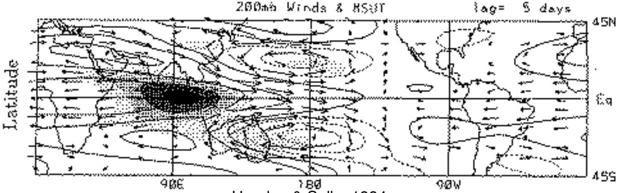
suppressed convection ($A < 0$)

enhanced convection ($A > 0$)

Ingredients:

- Kelvin wave structure on equator
- ***Rossby gyre structure off equator***

Observed MJO structure



Hendon & Salby 1994

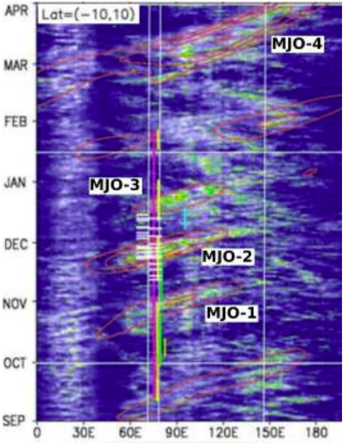
Motivation for Stochastic Skeleton Model

Need to capture:

1. intermittent generation of MJO events
2. organization of MJO events into wave trains (with growth and demise of wave trains)

Wave train of 2–3 MJO events ⟶

MJO events during
DYNAMO/CINDY 2011–2012



Yoneyama et al 2014

A Stochastic Skeleton Model for the MJO

Thual, Majda, & Stechmann 2014 *JAS*

Replace $\partial_t a = \Gamma q a$

with *stochastic jump process* for growth/decay of a,

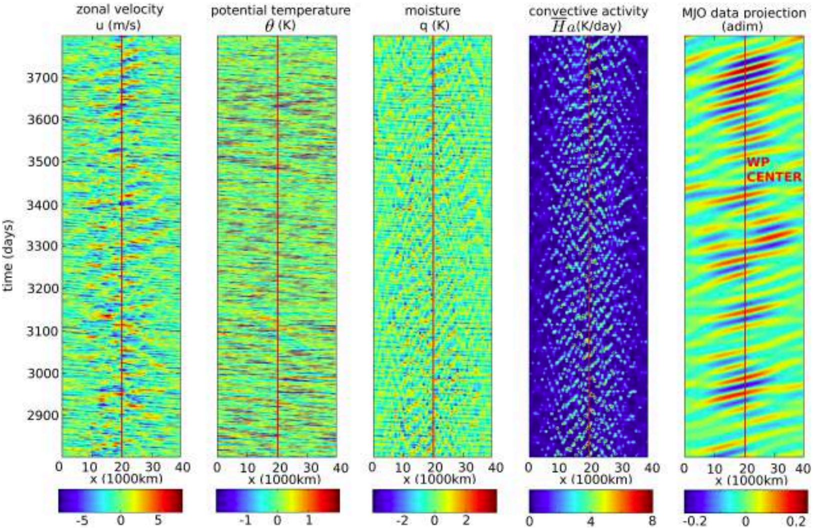
which satisfies $\partial_t \langle a \rangle = \Gamma \langle q a \rangle$ in the mean

Intuition:

Growth/decay of convective activity is *stochastic*,

due to unresolved synoptic/mesoscale fluctuations

Space-time variability



- intermittent generation of MJO events
- organization of MJO events into wave trains

(Geometric ergodicity, Majda and Xin Tong, CPAM 2015)

MJO event statistics in skeleton model and observations

(Stachnik, Waliser, Majda, Stechmann, Thual)

Number of MJO events:

Event Type	Observations 1979–2012	Stochastic Skeleton Model Idealized warm pool, 34 yrs
Primary	154	106
Continuing	330	381
Circumnavigating	15	27
Terminal	154	106

Average Duration of MJO events:

Observations: 39.7 days
Stochastic Skeleton Model: 34.8 days

Stochastic Skeleton Model reproduces *Observed MJO Statistics*

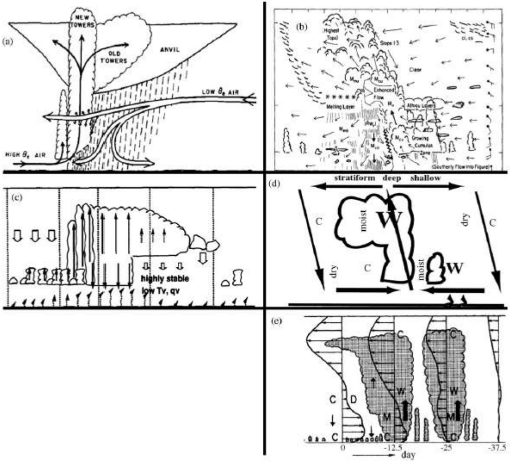
III. Hierarchy of Models: Multicloud Model in a GCM

Multiscale self-similar convective systems often
embedded in other like Russian dolls

Squall lines
(200 km)

C.C.W.
(2000 km)

MJO
(20, 000 km)



Compiled by Mapes et al. DAO. 2006

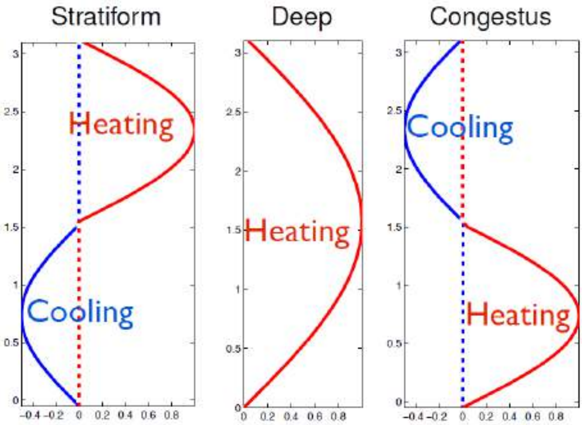
Why these three values for coherent structure? Majda, *JAS* 2007.

The Multicloud Model Dynamics

(Khouider and Majda, *JAS*, 2006, 2008, 2010)

- Based on three cloud types, congestus, deep, and stratiform.
- Moisture Switch: Dry lower troposphere favors congestus clouds while moist lower troposphere favors deep convection.
- Stratiform clouds lag deep convection.
- Associated heating profiles force the first two baroclinic modes of vertical structure.
- MC Model is coupled to the boundary layer and to a vertically averaged moisture equation through cold pool downdrafts and precipitation.
- Two shallow water equations with interactive source terms.

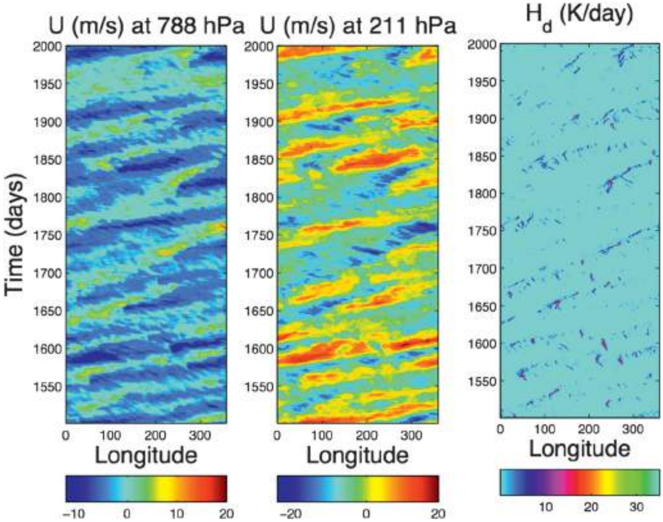
The Multicloud Model



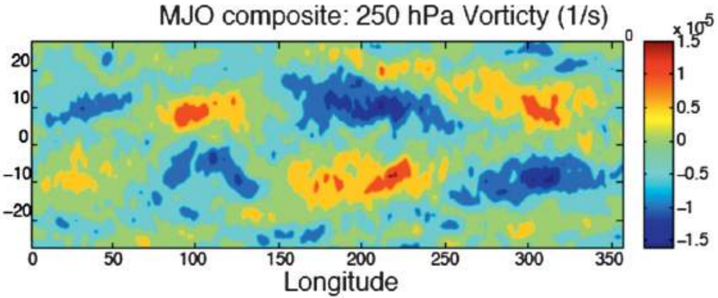
MJO in MC-HOMME

(Khouider, St-Cyr, Majda, Tribbia, *JAS* 2011)

Inexpensive coarse resolution ($\Delta x = 160$ km)



MC-HOMME MJO Structure



CPCM-NYUAD:

Many more realistic MJO and Monsoon scenarios with stochastic multi-cloud model and coarse resolution.

(Ajayamohan Ravindran, Khouider, Majda, Deng, 2013-2015)

Multi-scale models for tropics are rich source of physical phenomena, new equations and PDE problems.

Expository articles:

- R. Klein, "Scale-dependent models for atmospheric flows", *Ann. Rev. Fluid Mech.*, 2010
- B. Khouider, A. Majda, S. Stechmann, "Climate Science in the Tropics: Waves, Vortices, and PDEs", *Nonlinearity*, 2013

Simplified moisture models interacting with shear, waves and vortices

- Multiscale models for the hurricane embryo, Majda etal, *J. Fluid Mech.*, 2010
- Minimal models for precipitating turbulent convection, Hernandez-Duenas, Majda, Leslie Smith, Stechmann, *J. Fluid Mech.*, 2014

and MORE!!

Thank you