Generalized Smoluchowski Equations and Scalar Conservation Laws:

Random Solutions of $\quad u_t = H(x,t,u_x)$, $\quad H(x,t,p) = H(p) + V(x,t)$.

$$u_x = \rho \qquad \rho_t = \big(H(x,t,\rho)\big)_x$$

where V random in x,t.

There are nice conjectures about the law of ρ

when either $\begin{cases} H \text{ is convex} \\ V \equiv 0 \end{cases}$ or $\quad H(p) = \frac{1}{2}p^2$, V is Brownian in x, white noise in time

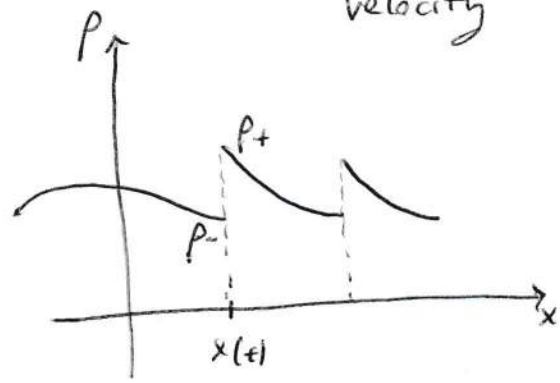
(Menon - Srinivasan) $\qquad$ (Chabanol - Duchon).

Some features of the PDE $\begin{cases} \rho_t = (H(\rho))_x, & H \text{ convex} \\ \rho_t = \underbrace{H'(\rho)}_{\text{velocity}}\, \rho_x \end{cases}$

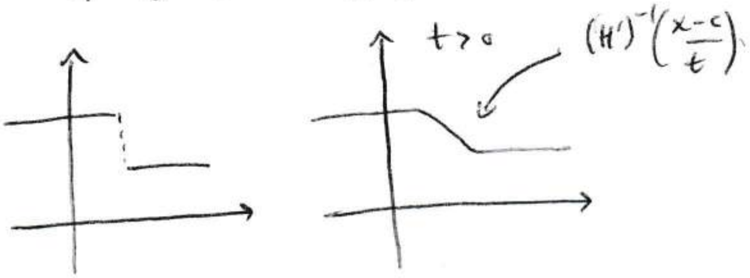
$$\frac{dx}{dt} = -H[\rho_-,\rho_+]$$

where

$$H[\rho_-,\rho_+] = \frac{H(\rho_-) - H(\rho_+)}{\rho_- - \rho_+}$$



If $t=0$ we have



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<u>Conjecture</u>: If $\rho^0(x)$ is a Markov process as a function of x initially, (with positive jumps)

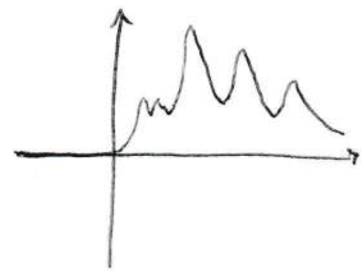
$\Longrightarrow$ Same is true at later times and we get a precise formula for the law.

CD worked this out for the case $\rho^0 =$ Lévy process.

<u>Theorem</u> (Bertoin 1998–2000) Assume $\rho^0(x) = \begin{cases} \text{Lévy with + jumps} & \text{for } x>0 \\ 0 & \text{for } x<0. \end{cases}$

Assume $\mathbb{E}\,\rho(1,0) \le 0$.

Then $\rho(x,t) - \rho(0,t)$ is a Lévy process.



Moreover, $\mathbb{E}\, e^{-\lambda(\rho(x,t)-\rho(0,t))} = e^{x\psi(\lambda,t)}$

$$\text{with} \quad \psi_t + \psi \cdot \psi_\lambda = 0.$$

In fact, if we write $\psi(\lambda,t) = \int_0^\infty (e^{-s\lambda} - s\lambda + 1)\,\nu_t(d\lambda)$

Then ν_t solves $\frac{d}{dt}\nu_t(dm) = \frac{m}{2}(\nu_t * \nu_t)(dm) - (M+m)\,\nu_t(dm)$

where $M = \int_0^\infty \lambda \cdot \nu_t(dm)$ is independent of time.

What if instead of Burgers' eqn we have $\rho_t = H(\rho)_x$

with H which is not quadratic?

According to Menon–Srinivasan Conjecture

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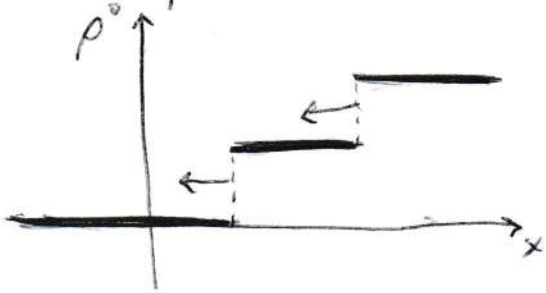
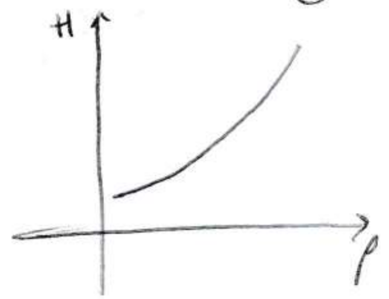
Markov process with positive jumps initially

$\Longrightarrow$ Markov process at later time

If $f(t, \rho_-, d\rho_+)$ then a Smoluchowski type eqn governs the evolution of f.

<u>Theorem</u> (with Kaspar) Assume H is convex and increasing. Let ρ^0 be as in the figure

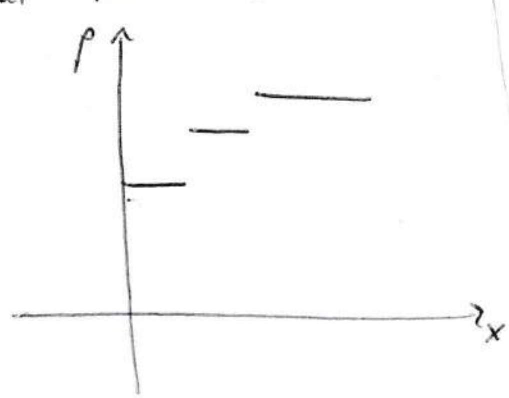




with jump rate given by $f^0(\rho_-, d\rho_+)$.

Further assume $\int_{\rho_-}^{\infty} f^0(\rho_-, d\rho_+) \equiv \lambda$, independent of ρ_-.

Then $\rho(x,t)$ is a jump process that looks like



with $\rho(x,t)$ a jump process with rate $f(t, \rho_-, d\rho_+)$ and $\rho(\cdot, 0)$ a jump process, also with a suitable rate.

there f solves

$$\frac{d}{dt} f(t, \rho_-, d\rho_+) = \int_{\rho_-}^{\rho_+} \left(H[\rho_*, \rho_+] - H[\rho_-, \rho_*]\right) f(t, \rho_-, \rho_*)\, f(t, \rho_+, d\rho_*)$$

(continued...)

$$- \left(A_f(\rho_-) - A_f(\rho_+)\right) f(t, \rho_-, d\rho_+)$$

where $A_f(a) = \int H[a, \rho_*] f(t, a, d\rho_*)$.

Moreover $\rho(0,t)$ is a jump process with jump rate given by

$$f(t, \rho_-, d\rho_+) H[\rho_-, \rho_+].$$

<u>Remark</u> To have global solution, one may assume that the support of $f^0(\rho_-, d\rho_+)$ is always a finite interval $[a,b]$ and that initially ρ^0 is always in this interval. Otherwise solution may blow-up in finite time.

Recall $$H[\rho_-, \rho_+] = \frac{H(\rho_-) - H(\rho_+)}{\rho_- - \rho_+}$$

<u>Idea of the proof</u>: If $x_1 < x_2 < x_3 < \cdots$ are the location of discontinuities and $\rho_1 < \rho_2 < \rho_3 < \cdots$ are the values of ρ. Solve the Smoluchowski equation, use the solution to build a candidate for the law of the configuration $(x_1, \ldots, \rho_1, \ldots)$ at time t. Then show that this law solves the "forward" equation associated with the deterministic evolution.