Kadomtsev-Petviashvili II in 2 and 3d

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The Kadomtsev-Petviashvili II equation

Consider

$$\partial_x(u_t + u_{xxx} + \partial_x u^2) + \Delta_y u = 0$$

$$t, x \in \mathbb{R}, y \in \mathbb{R}^d, \qquad d = 1, 2$$

1. Solutions: Line soliton (expected to be stable for KPII, unstable for KPI)
2. Symmetries:
   - Translation
   - Scaling: $\lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y)$. Fourier: $\lambda^{-d}\hat{u}(\lambda^{-3}\tau, \lambda^{-1}\xi, \lambda^{-2}\eta)$
   - Galilean symmetry: $u(t, x - v \cdot y - |v|^2 t, y + 2tv)$. Fourier: $\hat{u}(\tau - |v|^2\xi - 2v\eta, \xi, \eta + v\xi))$
3. Invariant function spaces: $\|u_0\|_{\dot{H}^{-1/2,0}} = \||\xi|^{-1/2}\hat{u}_0\|_{L^2}$ for $d = 1$ and $\|u_0\|_{\dot{H}^{1/2,0}} = \||\xi|^{1/2}\hat{u}_0\|_{L^2}$ for $d = 2$

Line soliton

Theorem (Hadac, Herr, Koch '09)

There exists $\varepsilon > 0$ *such that if* $\|u_0\|_{\dot{H}^{-1/2,0}} < \varepsilon$ *then there exists a unique solution* u *in a function space* X *which satisfies* $\|u\|_{C(\mathbb{R};\dot{H}^{\frac{1}{2},0})} \leq c\|u\|_X \leq c\|u_0\|_{\dot{H}^{-\frac{1}{2},0}}$. *The solution scatters.*

Theorem (Koch, Li '15)

There exists $\varepsilon > 0$ *such that if*

$$\|u_0\|_{\tilde{H}^{\frac{1}{2},0}} < \varepsilon$$

then there exists a unique solution $u = S(t)u_0 + v$ *with that initial data in a function space* X *which satisfies*

$$\|v\|_{C_b([0,\infty),\tilde{H}^{\frac{1}{2}})} \leq c\|v\|_X \leq c\|u_0\|^2_{\tilde{H}^{\frac{1}{2}}}.$$

Previous results

- 2d: Bourgain L^2, Takaoka& Tzvetkov, Isaza& Mejia, Takaoka, Hadac.
- 3d: Tzvetkov, Isaza, Lopez & Mejia, Hadac (almost invariant)
- First results in function spaces invariant under the symmetries of the problem
- Thesis Schottdorf '13: Systems of Klein-Gordon equations with quadratic nonlinearities.

A toy problem

Consider in $\mathbb{R} \times \mathbb{R}^2 \ni (t, x)$

$$i\partial_t u + \Delta u = \partial_{x_1} \bar{u}^2$$

with initial condition $u(0, x) = u_0(x)$.

Theorem

There exists $\varepsilon > 0$ such that for all u_0 with $\|u_0\|_{L^2} < \varepsilon$ there exists a unique global in time solution u. It scatters at ∞: The limit

$$\lim_{t \to \infty} e^{-it\Delta} u(t)$$

exists in L^2.

Step 1: Littlewood-Paley decomposition, duality

Let $\lambda \in 2^{\mathbb{Z}}$ and $\hat{u}_\lambda = \chi_{\lambda \le |\xi| < 2\lambda} \hat{u}$. Let

$$\|u\|_X = \left(\sum_{\lambda \in 2^{\mathbb{Z}}} \|u_\lambda\|_{V^2}^2 \right)^{1/2} .$$

Then

$$v(t) = \begin{cases} e^{it\Delta} u_0 & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

satisfies

$$\|v\|_X \le \sqrt{2} \|u_0\|_{L^2}$$

and

$$\left\| \int_0^t S(t-s) f(s) ds \right\|_{V^2} \le 2 \sup_{\|v\|_{V^2} \le 1} \left| \int f \bar{v} dx dt \right| .$$

Reduction to a trilinear estimate

We claim

$$\left| \int_{\mathbb{R}\times\mathbb{R}^2} \bar{u}\bar{v}\partial_{x_1}\bar{w}\,dxdt \right| \le c\|u\|_X\|v\|_X\|w\|_X. \tag{1}$$

Then, by duality

$$\left\| \int_0^t e^{i(t-s)\Delta}\partial_{x_1}\bar{u}\bar{v}\,ds \right\|_X \le c\|u\|_X\|v\|_X$$

and the theorem follows by standard arguments.

Littlewood-Paley reduction

We expand

$$u = \sum_{\lambda \in 2^{\mathbb{Z}}} u_\lambda$$

where

$$\hat{u}_\lambda = \chi_{\lambda \le |\xi| < 2\lambda} \hat{u}$$

and expand the integral. We claim

$$\sum_{\mu \le \lambda} \left| \int \bar{u}_\mu \bar{v}_\lambda \bar{w}_\lambda dxdt \right| \le c\lambda^{-1} \left(\sum_{\mu \le \lambda} \|u_\mu\|_{V^2}^2 \right)^{1/2} \|v_\lambda\|_{V^2} \|v_\lambda\|_{V^2}. \tag{2}$$

Dyadic implies full estimate

We expand (with sums over $2^{\mathbb{Z}}$)

$$\int \bar u \bar v \partial_{x_1} \bar w dx dt \le \sum_{\lambda_1,\lambda_2,\lambda_2} \left| \int \bar u_{\lambda_1} \bar v_{\lambda_2} \partial_{x_1} \bar w_{\lambda_3} . dx dt \right| .$$

Since the integral of the product is the evaluation of the Fourier transform of the triple convolution at 0, there is only a contribution if there are

$$\xi_1 + \xi_2 + \xi_3 = 0, \lambda_j \le |\xi_j| \le 2\lambda_j .$$

Then necessarily the two larger numbers of λ_j are of similar size. To simplify the notation we assume that they are equal and we denote them by λ and the smaller number by μ.
Moreover

$$\|\partial_{x_1} w_{\lambda_3}\|_{V^2} \le 2\lambda_3 \|w_{\lambda_3}\|_{V^2}$$

and we may replace the derivative with a multiplication by λ_3.

The bound

We bound using (2)

$$\sum_{\lambda}\sum_{\mu\leq\lambda}\left|\lambda\int \bar{u}_\mu\bar{v}_\lambda\bar{w}_\lambda dxdt\right| \leq c\sum_{\lambda}\left(\sum_{\mu\leq\lambda}\|u_\mu\|_{V^2}^2\right)^{1/2}\|u_\lambda\|_{V^2}\|w_\lambda\|_{V^2}$$
$$\leq c\|u\|_X\|v\|_X\|w\|_X$$

and

$$\sum_{\lambda}\sum_{\mu\leq\lambda}\mu\left|\int \bar{u}_\lambda\bar{v}_\lambda\bar{w}_\mu dxdt\right| \leq c\sum_{\lambda}\sum_{\mu\leq\lambda}\frac{\mu}{\lambda}\|u_\lambda\|_{V^2}\|v_\lambda\|_{V^2}\|w_\mu\|_{V^2}.$$
$$\leq c\|u\|_X\|v\|_X\|w\|_X$$

Function spaces I

The linear Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

has a fundamental solution

$$g_t(x) = ((4\pi i t)^{1/2})^{-n} e^{-\frac{|x|^2}{4it}}$$

with Fourier transform

$$\hat{g}_t(x) = e^{it|\xi|^2}$$

hence

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \qquad \|u(t)\|_{L^\infty} \leq |4\pi t|^{-n/2} \|u_0\|_{L^1}$$

It defines a unitary group $S(t)$ (Fourier transform). Solutions are called *free waves*.

Function spaces II

Bourgain:

$$\|u\|_{X^{s,b}} = \|S(-t)u(t)\|_{H^b(\mathbb{R};H^{s,0}(\mathbb{R}^{1+d}))} = \left\| |\xi|^s |\tau - |\xi|^2|^b \hat{u} \right\|_{L^2}$$

With $v(t) = S(-t)u(t)$ the problem becomes

$$v_t = S(-t)\partial_x (S(t)v(t))^2$$

Useful for scattering, better regularity properties, study of resonances.
For critical problems one would want to use $X^{s,1/2}$ and $X^{s,-1/2}$ - does usually not work due to failing imbeddings like $H^{1/2} \not\subset C(\mathbb{R})$, $L^1 \not\subset H^{-1/2}$.

Function spaces III

Replacement (Tataru, Koch and Tataru, Hadac & Herr & Koch)

$$B^{\frac{1}{2}}_{2,1} \subset U^2 \subset V^2_{rc} \subset B^{\frac{1}{2}}_{2,\infty}$$

Advantage: Functions in V^2_{rc} are bounded and

$$\|u\|_{U^2(\mathbb{R})} \le \sup \left\{ \int v u' dt : \|v\|_{V^2_{rc}} \le 1 \right\}$$

Probability and harmonic analysis: Brownian motion, Wiener, Lepingle, Bourgain, Lyons.
We define

$$\|u\|_{U^2_S} = \|S(-t)u\|_{U^2}$$

and similarly we deal with U^p and V^p.

Properties

- Duality
- High modulation
- Strichartz and dual Strichartz
- Bilinear estimates
- Scaling

Modulation

Step 2. We want to bound the left hand side of (2), in particular

$$\left| \int \bar{u}_\mu \bar{v}_\lambda \bar{w}_\lambda dx dt \right| = \left| \hat{u}_\mu * \hat{v}_\lambda * \hat{w}_\lambda(0) \right|.$$

The integral is zero unless there are points in the support which add up to 0. If $\tau_1 = |\xi_1|^2$ and $\tau_2 = |\xi_2|^2$ and $\tau_3 = -\tau_1 - \tau_2$ and $\xi_3 = -\xi_1 - \xi_2$ then

$$\tau_3 - |\xi_3|^2 = -|\xi_1|^2 - |\xi_2|^2 - |\xi_1 + \xi_2|^2$$

Thus, with $\mu \leq \lambda$, in

$$\int \bar{u}_\mu \bar{v}_\lambda \bar{w}_\lambda dx\, dt$$

at least one of the terms has high modulation - i.e. vertical distance $\lambda^2/3$ to the characteristic set, otherwise the integral is zero.

High modulation on low frequency

We denote this term by ^{h} and we have to bound

$$\left| \int \bar{u}_\mu^h \bar{v}_\lambda \bar{w}_\lambda dx dt \right| \leq \|u_\mu^h\|_{L^2} \|(v_\lambda w_\lambda)_\mu\|_{L^2}$$
$$\leq \lambda^{-1} \|u_\mu\|_{V^2} \|v_\lambda\|_{L^4} \|w_\lambda\|_{L^4}$$
$$\leq \lambda^{-1} \|u_\mu\|_{V^2} \|v_\lambda\|_{U^4} \|w_\lambda\|_{U^4}$$

This completes the estimate in this case since

$$\|v_\lambda\|_{U^4} \leq c \|v_\lambda\|_{V^2}$$

and

$$\left(\sum_{\mu \leq \lambda} \|(v_\lambda w_\lambda)_\mu\|_{L^2}^2 \right)^{1/2} \leq \|v_\lambda w_\lambda\|_{L^2}$$

Bilinear estimate

The Fourier transform of a solution to the linear equation with initial data u_0 is

$$2\pi \hat{u}_0(\xi)\delta_{\phi=0}$$

and the distance to Σ measures the deviation from being a solution.
The remaining product has to be estimated in L^2. We consider first free solutions resp. distributions supported on a surface:

$$\|\hat{u}_0\delta_\phi * \hat{v}_0\delta_\phi\|_{L^2} \leq C\|u_0\|_{L^2}\|v_0\|_{L^2}$$

where (dyadic localization)

$$C^2 = \sup_{(\tau_i=|\xi_i|^2)} \int \delta_{(\phi(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1),\phi(\tau-\tau_2,\xi-\xi_2,\eta-\eta_2))}$$

Consequence:

$$\|S(t)u_0 S(t)v_0\|_{L^2} \leq C\|u_0\|_{L^2}\|v_0\|_{L^2}.$$

Strichartz estimates and bilinear estimates

Strichartz estimates for free waves

$$\|u\|_{L^p_t L^q} \le c\|u(0)\|_{L^2}$$

imply

$$\|u\|_{L^p_t L^q} \le c\|u\|_{U^p}$$

Bilinear estimates for free solutions

$$\|S(t)u_{0,\mu}S(t)v_{0,\lambda}\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \le c(\mu/\lambda)^{1/2}\|u_{0,\mu}\|_{L^2(\mathbb{R}^2)}\|u_{0,\lambda}\|_{L^2(\mathbb{R}^2)}$$

imply

$$\|u_\mu v_\lambda\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \le c(\mu/\lambda)^{1/2}\|u_\mu\|_{U^2}\|u_\lambda\|_{U^2}.$$

Application to KPII

Variables: $(t,x,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$. Fourier variables (τ,ξ,η).
Denote by u_λ the function with Fourier transform

$$\hat{u}_\lambda = \begin{cases} \hat{u} & \lambda \le |\xi| < 2\lambda \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $d=1$ and $\mu \le \lambda$

$$\|u_\mu v_\lambda\|_{L^2} \le c(\mu/\lambda)^{1/2} \|u_\mu\|_{U_S^2} \|v_\lambda\|_{U_S^2} \tag{3}$$

and for $d=2$

$$\|u_\mu v_\lambda\|_{L^2} \le c\mu \|u_\mu\|_{U_S^2} \|v_\lambda\|_{U_S^2}. \tag{4}$$

The dispersion relation with $\phi(\xi,\eta) = \xi^3 - \eta^2/\xi$

$$\tau_1 + \tau_2 - \phi(\xi_1+\xi_2, \eta_1+\eta_2) = -\xi_1\xi_2(\xi_1+\xi_2)\left(3 + \frac{\left|\frac{\eta_2}{\xi_2} - \frac{\eta_1}{\xi_1}\right|^2}{|\xi_1+\xi_2|^2}\right).$$

Key argument for $d=1$

$$\left|\int u_\mu^{>\mu\lambda^2} v_\lambda w_\lambda dx\,dy\,dt\right| \le \|u_\mu^{>\mu\lambda^2}\|_{L^2}\|(v_\lambda w_\lambda)_\mu\|_{L^2}$$
$$\le c\mu^{-1/2}\lambda^{-1}\|u_\mu\|_{V^2}\|v_\lambda\|_{V^2}\|w_\lambda\|_{V^2}$$

$$\left|\int u_\mu v_\lambda^{>\mu\lambda^2} w_\lambda dx\,dy\,dt\right| \le \|v_\lambda^{>\mu\lambda^2}\|_{L^2}\|u_\mu w_\lambda\|_{L^2}$$
$$\le c(\mu/\lambda)^{1/2}\mu^{-1/2}\lambda^{-1}\|u_\mu\|_{U^2}\|v_\lambda\|_{V^2}\|w_\lambda\|_{U^2}$$

$$\left\|\int_0^t S(t-s)\partial_x(u_{<<\lambda}^{>|\xi|\lambda^2} v_\lambda)ds\right\|_{V^2} \le \sup_{\|w_\lambda\|_{V^2}\le 1}\sum_{\mu\le\lambda}\left|\int u_\mu^{>\mu\lambda^2} v_\lambda \partial_x w_\lambda dx\,dy\,dt\right|$$
$$\le c\left(\sum_{\mu\le\lambda}\mu^{-1}\|u_\mu\|_{V^2}^2\right)^{1/2}\|v_\lambda\|_{V^2}$$

An almost proof for $d = 2$

$$\|u\|_X = \sum_\lambda \lambda^{1/2}\|u_\lambda\|_{V^2} + \lambda^{-1}\|u_\lambda\|_{X^{0,1}}$$

$$\|u_\mu v_\lambda\|_{L^2} \le \mu\|u_\mu\|_{U^2}\|v_\lambda\|_{U^2}$$

hence

$$\mu^{-1}\left\|\int_0^t S(t-s)\partial_x(u_\lambda v_\lambda)_\mu ds\right\|_{\dot{X}^{0,1}} \le c(\lambda^{1/2}\|u_\lambda\|_{V^2})(\lambda^{1/2}\|v_\lambda\|_{V^2})$$

(high,high to (low;high modulation))

$$\lambda^{1/2}\left\|\int_0^t S(t-s)\partial_x(u_\mu^{>\mu\lambda^2} v_\lambda)ds\right\|_{V^2} \le c\lambda^{5/2}\mu^{-1}\lambda^{-2}\|u_\mu\|_{X^{0,1}}\|v_\lambda\|_{U^2}$$

$$\le c\mu^{-1}\|u_\mu\|_{X^{0,1}}\lambda^{1/2}\|v_\lambda\|_{U^2}$$

((low;high modulaton), high to high) tight!

Problems

1. Replace U^2 by V^2
2. Summation
3. Initial data

Difficulty: Two of the four estimates are very tight. This can sometimes be handled (wave maps, Schrödinger maps) and sometimes not (Derivative NLS). No general recipe!

Improved bilinear estimate

Proposition

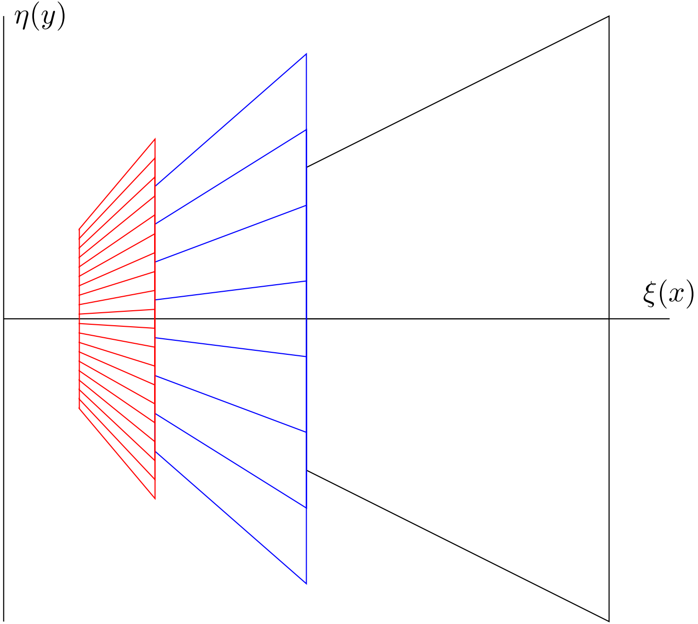
Let $A \subset \mathbb{R}^2$ and suppose that $u_{\mu,A}$ has a Fourier transform supported in $\mu \le |\xi| \le 2\mu$, $\frac{\eta}{\xi} \in A$. Then, if $\mu \le \lambda/8$,

$$\left\| \Big(\lambda + \Big|\frac{\eta_2}{\xi_2} - \frac{\eta_1}{\xi_1}\Big|\Big) S(t)u_{\mu,A} S(t)v_\lambda \right\|_{L^2} \le c\mu |A|^{1/2} \|u_{\mu,A}\|_{L^2} \|u_\lambda\|_{L^2}$$

This is reminiscent of the bilinear estimate of the Korteweg-de-Vries equation.

Transformation formula, fix $\frac{\eta-\eta_2}{\xi-\xi_2} = \rho \in A$, and consider the integral with ξ for fixed ρ. This is a one dimensional bilinear estimate, like for KdV.

The geometry



Function spaces

Let $p < 2$.

$$\|u\|_{l^1 l^p V^2} = \sum_\lambda \lambda^{1/2} \|u_\lambda(\lambda x, \lambda^2 y, \lambda^3 t)\|_{l^p V^2}$$

where

$$\|u_1\|_{l^p V^2}^p = \sum_j \|u_{\Gamma_{j,1}}\|_{V^2}^p$$

and for $j \in \mathbb{Z}^2$,

$$\Gamma_{j,1} = \left\{ (\xi, \eta) : 1 \le |\xi| \le 2, j - \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \le \eta/\xi < j + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \right\}$$

and, with some $\frac{5}{6} < b < 1$

$$\|u\|_X = \||D_x|^{1/2} u\|_{l^1 l^p V^2} + \|u\|_{l^1 l^p X^{2-3b,b}}.$$

Analogue decomposition of initial data.