The Kardar-Parisi-Zhang equation and universality class:

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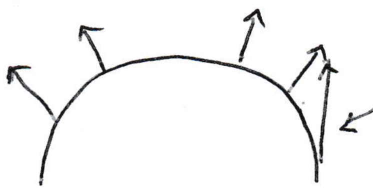
KPZ equation

$$\partial_t h = \underbrace{(\partial_x h)^2}_{\text{lateral growth}} + \underbrace{\partial_x^2 h}_{\text{relaxation term}} + \xi$$

space-time white noise (random forcing).

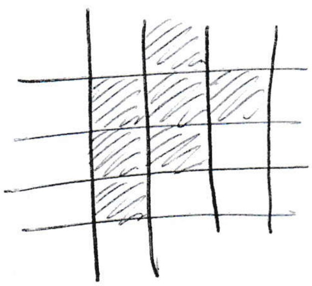


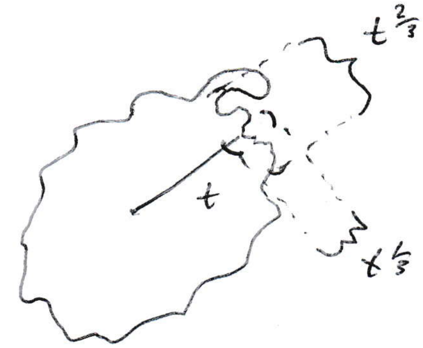
$$F(\partial_x h) = F(0) + F'(0)\partial_x h + \frac{1}{2}F''(0)(\partial_x h)^2 + \cdots$$

first term that cannot be removed.

Model (1) Random growth

E.g. (a) Eden Model





no results of any sort.

(b) ASEP

$q + p = 1$

$q > p$

invariant measure: simple random walk.

$\downarrow$

invariant measure of KPZ: Brownian motion.

(b) continued...

very solvable if $q=1, p=0$ TASEP

somewhat solvable if $q>p$

KPZ appear in limit $q \sim p$.

② Directed random polymer free energies.

③ $\partial_x h = u$

$$\partial_t u = \partial_x (u^2) + \partial_x^2 u + \partial_x \xi$$

stochastic Burgers eqn.

invariant measure: spatial white noise.

Dimensions $d>1$:

(*) $\partial_t u = \nabla \cdot (F(u) + D \nabla u + \xi)$ white noise invariant.

$\Longrightarrow \partial_t h = |\nabla h|^2 + \Delta h + \xi$ invariant measure unknown except when $d=1$.

(In fact, nothing is known).

From (*), "compute" $S(x,t) = \langle u(x,t) ; u(0,0) \rangle_{eq}$

Is it diffusive? Answer: $\underbrace{\text{Yes}}_{\text{proved in certain particle models.}}$ in $d \geq 3$, No in $d=1,2$

$v = F'(\rho)$

$$\int (x - vt)^2 S(x,t)\,dx \overset{\text{conj}}{=} t \quad \text{diffusive} \quad d \geq 3$$

$$\overset{\text{conj}}{\sim} t\sqrt{\log t} \quad d = 2$$

$$\boxed{\text{THM Valko \& Q.} \quad \log\log t \leq \underbrace{}_{} \leq \log t}$$

$$\overset{\text{conj}}{\sim} t^{4/3} \quad d = 1.$$

<u>$d=1$</u> $S(x,t) \overset{\text{conj}}{\sim} c_1 t^{-2/3} f_{KPZ}(t^{-2/3}x)$

(known in one case only)

full of Airy functions and Fredholm determinants.

Why should YOU care?

E.g. $H_N = \sum \frac{p_i^2}{2} + V(\overbrace{q_{j+1} - q_j}^{r_j})$ (non-integrable)

think of

$$V_{FPU} = \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}.$$

conserved

$\sum r_j, \sum p_j, \sum \frac{p_j^2}{2} + V(r_j)$ (non-integrable = only these cons. laws)

Let $\vec{u} = (r, p, e) = (u_1, u_2, u_3)$

$S_{ij}(x,t) = \langle u_i(x,t), u_j(0,0) \rangle$

conj f_{KPZ}

$S_{11}(x,t)$

very good fit in computer simulation.

Back to KPZ

<u>Well-posedness</u> $h = \log Z$, $\partial_t Z = \partial_x^2 Z + \xi Z$.

defn

gives "TRUE KPZ" $\partial_t h = ((\partial_x h)^2 - \infty) + \partial_x^2 h + \xi$.

see Hairer, Gubinelli, Perkowski, Imkeller,

<u>Large space-time</u> $h_\varepsilon(t,x) = \varepsilon^b h(\varepsilon^{-z}t, \varepsilon^{-1}x)$

$$\partial_t h_\varepsilon = \varepsilon^{2-z-b}(\partial_x h_\varepsilon)^2 + \varepsilon^{2-z}\partial_x^2 h_\varepsilon + \varepsilon^{b-\frac{1}{2}z+\frac{1}{2}}\xi.$$

$b = \frac{1}{2} \Rightarrow z = \frac{3}{2} \Rightarrow$ fluctuation $\sim t^{1/3}$

spatial scale $\sim t^{2/3}$

diffusivity $\sim t^{4/3}$.

(since B.M is invariant)

$$\partial_t h_\varepsilon = (\partial_x h_\varepsilon)^2 + \varepsilon^{1/2}\partial_x^2 h_\varepsilon + \varepsilon^{1/4}\xi.$$

Limit is NOT inviscid Burgers

"Limit" is the KPZ fixed point.

<u>Weakly asymmetric limit</u> start with asymmetry $\varepsilon^{1/2}$

rescale diffusivity $z=2 \Rightarrow$ KPZ

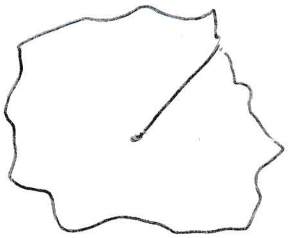
E.g. $\varepsilon^{1/2} h^{ASEP}_{q-p=\varepsilon^{1/2}}(\varepsilon^{-2}t, \varepsilon^{-1}x) - C(\varepsilon,t) \longrightarrow$ KPZ eqn.

Thm Bertini-Giacomin

Quastel

(5)

Large scale fluctuations depend on initial data (class)

Curved  $Z_0(x) = \delta_0(x), \quad h_0(x) = \log \delta_0(x)$

"narrow wedge"

long-time limit

$$h(t,x) \sim c_1 t + c_2 \frac{x^2}{t} + c_3 t^{1/3} \zeta_{GUE}$$

$$P(\zeta_{GUE} \le s) = F_{GUE}(s) = \det(I - P_0 K_s P_0) \quad \left(\begin{array}{c}\text{this is a } \underline{\text{THM}} \\ \text{for KPZ}\end{array}\right).$$

projection onto $[-\infty, s]$

eigenspace of $H = -d_x^2 + x$

Airy operator.

GUE matrix

$N \times N, \quad a_{ij} = \overline{a_{ji}}$

i.i.d. complex Gaussian,

mean zero, variance N

largest e. value $\lambda_N = N + N^{1/3} \zeta_{GUE}$.

Flat $Z_0(x) = 1, \quad h_0(x) = 0$

$$h(t,x) \sim c_1 t + c_2 t^{1/3} \zeta_{GOE}$$

orthogonal

still technical problems

Way 1st one is proved is ① Exact formula h^{ASEP} (Tracy-Widom)

② Use weakly asymmetric limit + steepest descent to get a formula for KPZ.

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