Martel: Supplementary Boardwork.

- $C([0,T), H^1(\mathbb{R}))$, $T \le \infty$

$\boxed{T < \infty}$ $\quad \lim_{t \uparrow T} \|\nabla u(t)\|_{L^2} = +\infty$.

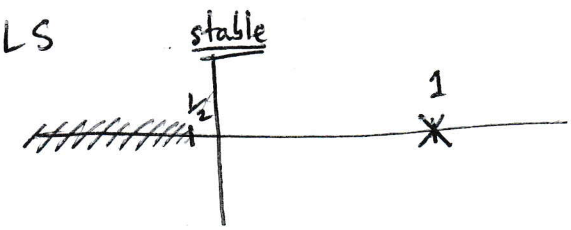
NLS $\quad \|\nabla u(t)\|_{L^2} \gtrsim \dfrac{1}{\sqrt{T-t}}$

gKdV $\quad \|\partial_x u(t)\|_{L^2} \gtrsim \dfrac{1}{(T-t)^{1/3}}$

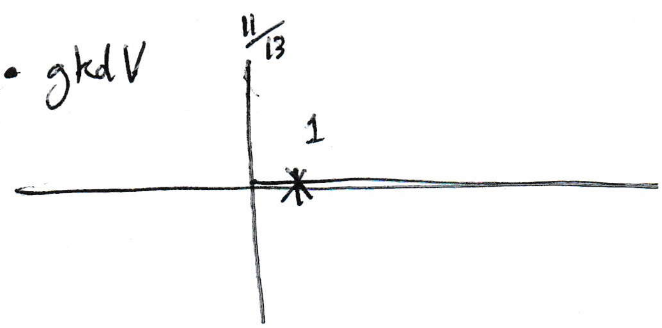
$$\|\partial_x u(t)\|_{L^2} \sim \frac{1}{\lambda_0}$$

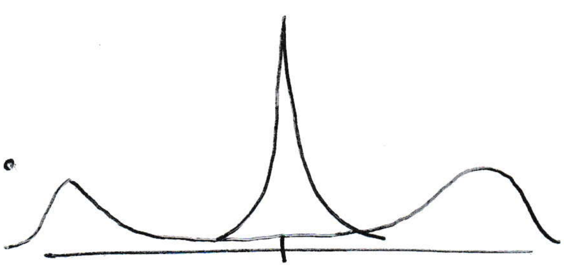
$$\|\nabla S_{NLS}(t)\|_{L^2} \sim \frac{1}{(T-t)}$$

- NLS



- gKdV



- 

$$u(t,x) = \frac{1}{\lambda^{1/2}(t)} Q\left(\frac{x - y(t)}{\lambda(t)}\right) + \text{ERROR}.$$

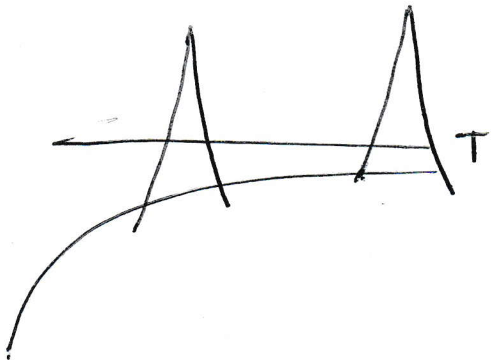
$\lambda'' = 0$ $\quad$ { control in weighted space $\mathcal{A}$

- **STABLE:**

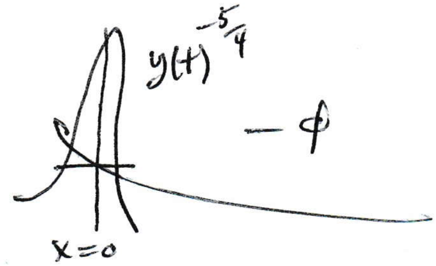
$\lambda' = l_0$ $\quad \begin{cases} E \le 0 \Rightarrow l_0 < 0 \\ \lambda(t) = l_0 (T-t) \end{cases}$

Martel

$$y'(t) = \lambda^{-2}(t), \quad y(t) = \frac{1}{(T-t)}$$



Exotic Blow-up: $\phi(x) = x^{-5/4}, \ x \gg 1$



$$(Q_\lambda)_t + ((Q_\lambda)_{xx} + (Q_\lambda + \phi)^5)_x = 0$$

$$5(\phi Q_\lambda^4)_x$$

$$Q_\lambda + \boxed{\phi}$$

$$\left(\lambda' + \lambda^{-3/2} y^{-5/4}(t)\right)' = 0 \ ; \quad y' = \lambda^{-2}$$

$$\lambda' + \lambda^{-3/2} y^{-5/4} = 0 \qquad \lambda^{1/2} - y^{-1/4} \qquad \boxed{e^{t/2}}$$

$$\lambda^{-1/2}\lambda' + y' y^{-5/4} = 0 \qquad \boxed{x^2 = y^{-1}}$$

$$\boxed{y' = y} \qquad \boxed{e^t}$$

Exotic blow up rates for some critical nonlinear dispersive equations

Yvan Martel

Ecole Polytechnique & MSRI (Fall 2015)

Two mass critical (focusing) models

- Quintic gKdV equation

$$\begin{cases} \partial_t u + \partial_x(\partial_x^2 u + u^5) = 0 \\ u_{|t=0} = u_0 \in H^1 \end{cases} \qquad (t,x) \in [0,T) \times \mathbb{R}$$

- Cubic NLS in 2D

$$\begin{cases} i\partial_t u + \Delta u + |u|^2 u = 0 \\ u_{|t=0} = u_0 \in H^1 \end{cases} \qquad (t,x) \in [0,T) \times \mathbb{R}^2$$

The CP for both models is well-posed in H^1 locally in time [Kenig-Ponce-Vega (1993)], [Ginibre-Velo (1979))]

This talk is concerned with finite time blow up solutions

$$\lim_{t\uparrow T} \|\nabla u(t)\|_{L^2} = +\infty \qquad 0 < T < +\infty$$

gKdV

- Solitons are traveling wave solutions of gKdV

$$u(t,x) = \frac{1}{\lambda_0^{\frac{1}{2}}} Q_{\mathrm{KdV}}\left(\frac{1}{\lambda_0}(x-x_0) - \frac{1}{\lambda_0^3} t\right), \quad \lambda_0 > 0,\ x_0 \in \mathbb{R}$$

$$Q_{\mathrm{KdV}}(x) = \left(\frac{3}{\cosh^2(2x)}\right)^{1/4}, \quad Q''_{\mathrm{KdV}} - Q_{\mathrm{KdV}} + Q^5_{\mathrm{KdV}} = 0$$

- Mass and energy conservation

$$M_{\mathrm{KdV}}(u(t)) = \int u^2(t), \quad E_{\mathrm{KdV}}(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{6}\int u^6(t)$$

- Global existence for mass below the mass of Q_{KdV}
[Weinstein, 1983]

$$\|u_0\|_{L^2} < \|Q_{\mathrm{KdV}}\|_{L^2} \quad \Rightarrow \quad \text{the solution is global in } H^1$$

NLS

- Solitons are solitary wave solutions of NLS

$$u(t,x) = \frac{e^{i\frac{t}{\lambda_0^2}}}{\lambda_0} Q_{\mathrm{NLS}}\left(\frac{1}{\lambda_0}(x-x_0)\right), \quad \lambda_0 > 0,\ x_0 \in \mathbb{R}^2$$

$$\Delta Q_{\mathrm{NLS}} - Q_{\mathrm{NLS}} + Q_{\mathrm{NLS}}^3 = 0$$

- Mass and energy conservation

$$M_{\mathrm{NLS}}(u(t)) = \int |u|^2(t), \quad E_{\mathrm{NLS}}(u(t)) = \frac{1}{2}\int |\nabla u|^2(t) - \frac{1}{4}\int |u|^4(t)$$

- Global existence and scattering for mass below the mass of Q_{NLS} [Weinstein, 1983], [Killip-Tao-Visan, 2009], [Dodson, 2011]

- Explicit blow up solution with $\|S_{\mathrm{NLS}}\|_{L^2} = \|Q_{\mathrm{NLS}}\|_{L^2}$

$$S_{\mathrm{NLS}}(t,x) = \frac{1}{T-t} e^{-\frac{i|x|^2}{4(T-t)} + \frac{i}{T-t}} Q_{\mathrm{NLS}}\left(\frac{x}{T-t}\right)$$

Unique up to the symmetries of the equation [Merle, 1993]

Stable log-log blow up for NLS

Mass slightly above the mass of Q

$$\|Q_{\mathrm{NLS}}\|_{L^2} < \|u_0\|_{L^2} < \|Q_{\mathrm{NLS}}\|_{L^2} + \alpha, \quad 0 < \alpha \ll 1$$

• Existence of an open set in H^1 of blow up solutions with log-log speed (including negative and zero energy solutions)

$$\|\nabla u(t)\|_{L^2} \sim C^* \sqrt{\frac{\log|\log(T-t)|}{T-t}}$$

[Merle-Raphaël, 2003-2006], [Perelman, 2001], [Landman-Papanicolaou-Sulem-Sulem, 1988]

At blow up time $t \uparrow T$,

$$|u(t)|^2 \rightharpoonup \|Q_{\mathrm{NLS}}\|_{L^2}^2 \delta_{x=x(T)} + |u^*|^2, \quad u^* \in L^2$$

More pseudo conformal blow up and the Gap Theorem

- Unstable "Bourgain-Wang" blow up solutions $\frac{1}{T-t}$
[Bourgain-Wang, 1997],
[Krieger-Schlag, 2009], [Merle-Raphaël-Szeftel, 2011]

- Gap Theorem
A finite time blow up solution to NLS with mass slightly above the mass of Q satisfies one of the following
  - either log-log blow up
$$\|\nabla u(t)\|_{L^2} \sim C^* \sqrt{\frac{\log|\log(T-t)|}{T-t}}$$
  - or
$$\|\nabla u(t)\|_{L^2} \gtrsim \frac{1}{T-t}$$

[Raphaël, 2005]

Two results for perturbed NLS models

- Perturbation by a potential on the nonlinear term

Existence, uniqueness of minimal mass blow up solution (at 0) of

$$i\partial_t u + \Delta u + k(x)|u|^2 u = 0, \quad k(0) = 1, \quad \nabla^2 k(0) < 0$$

with

$$\|\nabla u(t)\|_{H^1} \sim \frac{1}{T - t}$$

[Raphaël-Szeftel, 2011]

- Perturbation by a lower order nonlinear term

Existence of a minimal mass blow up solution of

$$i\partial_t u + \Delta u + |u|^p u + |u|^2 u = 0 \quad 0 < p < 2$$

with

$$\|\nabla u(t)\|_{H^1} \sim (T - t)^{-\frac{2}{2+p}}$$

[Le Coz-M.-Raphaël, 2014]

Large mass blow up solutions for NLS

• K-point blow up. Weak interaction cases

Existence of solutions blowing up with conformal blow up rate at $K \geq 2$ given points. [Merle, 1990]

With log-log blow up rate [Planchon-Raphaël, 2007], [Chenjie Fan, 2015]

• Existence of "exotic" blow up solutions with strongly interacting solitary waves

Let $K \geq 2$. There exists a solution S_K of NLS such that

$$\|\nabla S_K(t)\|_{L^2} \sim \frac{|\log(T-t)|}{T-t}$$

and

$$|S_K(t)|^2 \rightharpoonup K\|Q\|_{L^2}^2 \delta_{x=0} \ \text{ as } t \uparrow T$$

[M.-Raphaël, 2015]

Stable blow up for gKdV

$$\mathcal{A} = \left\{ u_0 = Q + \varepsilon_0 \text{ with } \|\varepsilon_0\|_{H^1} < \alpha_0 \text{ and } \int_{x>0} x^{10}\varepsilon_0^2(x)dx < 1 \right\}$$

• Existence of an open subset of $\mathcal{A}$ of blow up solutions (including negative and zero energy solutions)

$$\|\partial_x u(t)\|_{L^2} \sim \frac{1}{T-t}$$

• $1/(T-t)$ is the only possible blow up rate close to Q for initial data in $\mathcal{A}$

[M.-Merle, 2000-2002], [M.-Merle-Raphaël, 2012]

Exotic blow up for gKdV close to Q

Construction of blow up solutions with exotic blow up rates using slow decay of the initial data

- Blow up in finite time, for any $\beta > \frac{11}{13}$,

$$\|\partial_x u(t)\|_{L^2} \sim (T-t)^{-\beta} \quad \text{as} \quad t \uparrow T.$$

- Blow up in infinite time:

$$\|\partial_x u(t)\|_{L^2} \sim e^t \quad \text{as} \quad t \to +\infty.$$

For any $\gamma > 0$,

$$\|\partial_x u(t)\|_{L^2} \sim t^\gamma \quad \text{as} \quad t \to +\infty.$$

[M.-Merle-Raphaël, 2012]

Energy critical models

- Wave map problem

- Stable regime [Rodnianski-Sterbenz, 2010], [Raphaël-Rodnianski, 2012]

- Unstable regime [Krieger-Schlag-Tataru, 2008]

- Schrödinger map system
[Bejenaru-Tataru, 2010], [Perelman, 2012], [Merle-Raphaël-Rodnianski, 2011]

- Energy critical wave
[Krieger-Schlag-Tataru, 2009], [Duyckaerts-Kenig-Merle, 2011], [Hillairet-Raphaël, 2012], [Jendrej, 2015]

- Energy critical NLS
[Ortoleva-Perelman, 2012]