Invariant measures and the soliton resolution conjecture

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The focusing nonlinear Schrödinger equation

- A complex-valued function u of two variables x and t, where $x \in \mathbb{R}^d$ is the space variable and $t \in \mathbb{R}$ is the time variable, is said to satisfy a d-dimensional focusing nonlinear Schrödinger equation (NLS) with nonlinearity parameter p if

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- The equation is called "defocusing" if the term $-|u|^{p-1}u$ is replaced by $+|u|^{p-1}u$. If the nonlinear term is absent, we get the ordinary Schrödinger equation.

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- That is, if $u(x,t)$ is a solution of the NLS, then $M(u(\cdot,t))$ and $H(u(\cdot,t))$ remain constant over time.
- List of all conserved quantities unknown (except in $d=1$, $p=3$).

Equivalence classes

- We will say that two functions u and v from $\mathbb{R}^d$ into $\mathbb{R}$ are equivalent if
$$v(x) = u(x - x_0)e^{\mathrm{i}\lambda_0}$$
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- Note that if u and v are equivalent in this sense, then $M(u) = M(v)$ and $H(u) = H(v)$.

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- Often, the function v is also called a soliton.

Ground state solitons

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  - For each $m > 0$, there is a unique $\lambda(m) > 0$ such that $Q_{\lambda(m)}$ is the ground state soliton of mass m.

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- The claim may not hold for all initial conditions, but is expected to hold for "generic" initial data.
- Significant progress in recent years (Kenig, Merle, Schlag, Tao, many others ....) but complete solution is still elusive.

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- Birkhoff's ergodic theorem: If μ is an ergodic invariant measure for the flow $\{T_t\}_{t\geq 0}$, then

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- Suggests that one can study the time-averaged long-term behavior of a flow by studying the space-average over an ergodic invariant measure μ.
- Easier to construct invariant measures than proving ergodicity.
- However, any invariant measure decomposes into ergodic components. Therefore Birkhoff's theorem implies that a high probability event for μ occurs within most ergodic components.

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- The focus in the PDE community has mainly been on using invariant measures to prove existence of global solutions.
- As a probabilist, my interest is more in understanding the space-average with respect to these invariant measures, and then appealing to the ergodic hypothesis.

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- This talk is based on:

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$$H(u) := \frac{h^d}{2} \sum_{x,y \text{ nhbrs}} \left|\frac{u(x)-u(y)}{h}\right|^2 - \frac{h^d}{p+1} \sum_x |u(x)|^{p+1}.$$

Microcanonical ensemble

- Fixing $\epsilon > 0$, $E \in \mathbb{R}$ and $m > 0$, define

$$S_{\epsilon,h,n}(E,m) := \{u : |M(u) - m| \le \epsilon, |H(u) - E| \le \epsilon\}.$$

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- A high probability event for f reflects the long-term behavior of the DNLS flow in "most" ergodic components of this invariant measure.
- Main question: What is the behavior of f? Does it look like a soliton in some limit?

Proof of a statistical version of the soliton resolution conjecture for the discrete NLS

Theorem (C., 2014)

Suppose that $1 < p < 1 + 4/d$*. Fix* E *and* m *such that* $E > E_{\min}(m)$*. Let* Q *be the ground state soliton of mass* m*. Let* f *be a uniform random choice from the set* $S_{\epsilon,h,n}(E, m)$*. Then for any* $\delta > 0$*,*

$$\lim_{h\to 0} \limsup_{\epsilon\to 0} \limsup_{n\to\infty} \mathbb{P}\left(\inf_{\substack{x_0\in\mathbb{R}^d \\ \lambda_0\in\mathbb{R}}} \max_{x\in h\mathbb{T}_n^d} |f(x) - Q(x-x_0)e^{\mathrm{i}\lambda_0}| > \delta \right) = 0\,.$$

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Important note: It is guaranteed by construction that $M(f) \approx m$ and $H(f) \approx E > E_{\min}(m)$.

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Important note: It is guaranteed by construction that $M(f) \approx m$ and $H(f) \approx E > E_{\min}(m)$. So f cannot be close to the ground state soliton in the H^1 norm. The theorem says that in an appropriate limit, f is close to the ground state soliton in the L^∞ norm.

Proof sketch

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Convergence to discrete solitons

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- Later, it is shown that $m^* \to m$ as $h \to 0$.

Convergence to discrete solitons (contd.)

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- By the stability of discrete solitons, suffices to prove that with high probability, $M(f^v) \approx m^*$ and $H(f^v) \approx E^*$ for some $m^* \in [0, m]$ and $E^* = E_{\min}(m^*, h)$.

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- Need upper bound on $V(m', E')$ and lower bound on V (that should actually closely match the true values).

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- The lower bound on V is obtained by guessing m^* and E^* and then using $V \geq V(m^*, E^*) \geq$ the volume of a neighborhood of a discrete soliton of mass m^* and energy E^*.

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- Lastly, observe that

$$|U| = |\{x : |f(x)| > \delta\}| \leq \delta^{-2} \sum |u(x)|^2 \approx \delta^{-2} m\,.$$

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- Thus,

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Let ξ be a random function chosen uniformly from the set of all functions $u : \mathbb{T}_n^d \to \mathbb{C}$ that satisfy $\sum |u(x)|^2 = 1$. Then for any $\alpha \in (0, 2d)$,

$$\lim_{n\to\infty} \frac{1}{n^d} \log \mathbb{P}\biggl(\sum_{x,\, y \;\, nhbrs} |\xi(x) - \xi(y)|^2 \le \alpha \biggr) = -\Psi_d(\alpha)\,,$$

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When $\alpha > 2d$, the same result holds after replacing $\le$ by $\ge$.

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That is all. Thanks for your attention.