Random versus Deterministic Approach in the Study of Wave and Dispersive Equations

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Introduction

- Lots of progress in the last 20 years in the study of nonlinear dispersive and wave equations.
- The thrust of this body of work has focused on deterministic aspects of wave phenomena.
- Yet there remain some important open questions especially in the supercritical case.

Consider for example the Cauchy IVP for the p-NLS equation:

$$\begin{cases} iu_t + \Delta u = \pm |u|^{p-1} u, \\ u(x,0) = u_0(x) \in H^s \qquad x \in \mathbb{R}^n \text{ or } \mathbb{T}^n \end{cases}$$

Recall: the scale invariant norm is $s_c := \frac{n}{2} - \frac{2}{(p-1)}$.

H^s data with $s > s_c$ is subcritical; $s = s_c$ is critical; $s < s_c$ is supercritical.

p-NLS: Deterministic GWP Results on $\mathbb{R}^n$

- **Critical Data Results** :
  - Global well-posedness and scattering for energy-critical NLS in $\mathbb{R}^n$
    - *Defocusing*: Bourgain; Grillakis; Colliander-Keel-S.-Takaoka-Tao; Killip-Visan, X. Zhang, Dodson.
    - *Focusing*: Kenig-Merle (concentrated compactness /rigidity method) and Killip-Visan.
  - Global well-posedness and scattering for mass-critical NLS in $\mathbb{R}^n$
    - *radial*: Killip, Tao, Visan, X. Zhang.
    - *nonradial*: Dodson
  - Global well-posedness and scattering for 'energy-supercritical' ($s_c > 1$) defocusing NLW and NLS **under the assumption of a uniform in time bound on the scale invariant norm** by Kenig and Merle; Killip and Visan; Bulut.
    - In spirit of Escauriaza, Seregin and Sverak for the Navier-Stokes equation.
- **Supercritical Data Results:** (?)

p-NLS: Deterministic GWP Results on $\mathbb{T}^n$

- **Critical Data Results**:
  - Global well-posedness for energy-critical NLS
    - *Defocusing and* $n = 3$: Ionescu-Pausader (large data, based on a work by Ionescu-Pausader-S.); and previously Herr-Tzvetkov-Tataru (small data).
  - Global well-posedness for mass-critical NLS
    - (?) *In fact there are no even local results at the* L^2 *level!*

Deterministic → Nondeterministic Approach

Bourgain considered the L^2-critical[1]:

Theorem (Rational Torus; Bourgain(96'))

$$\begin{cases} iu_t + \Delta u = |u|^2 u - \left(\int |u|^2 dx\right) u \\ u(x,0) = u_0(x), \quad x \in \mathbb{T}^2 \quad \textbf{(Rational)}, \end{cases}$$

is almost sure globally well-posed **below** L^2*; i.e. for* **supercritical data** $u_0 \in H^{-\varepsilon}$.

Very informal definition of almost sure well-posedness

Given μ a probability measure on the space of initial data X (eg. $X = H^s$)

There exists $Y \subset X$, with $\mu(Y) = 1$ and such that for any $u_0 \in Y$ there exist $T > 0$ and a unique solution u to the IVP in $C([0, T], X)$ that is also stable in the appropriate topology.

[1] In 93' Bourgain had proved LWP for $s > 0$ and GWP in $H^1(\mathbb{T}^2)$ for cubic NLS

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Bourgain's interest was to construct an **invariant Gibbs measure** derived from the PDE above viewed as an infinite dimension Hamiltonian system[2]:

1. Established local well posedness for 'typical elements' in the support of the measure; i.e. for random data ϕ^ω in $H^{-\varepsilon}(\mathbb{T}^2)$, (an 'almost sure' -in the sense of probability- LWP in $H^{-\varepsilon}(\mathbb{T}^2)$).
2. Proved that the associated Gibbs measure is **invariant** and used it to extend the local result to a global one in the almost sure sense.

Furthermore, Bourgain showed that almost surely in ω the nonlinear part of the solution

$$w := u - S(t)\phi^\omega$$

is smoother than the linear part.

Note: *This result still keeps open the question of (a.s.) global well-posedness when data are in H^s, $0 \leq s < 1$ since there are no invariant measures and no conservation laws. More later*.

[2] After Lebowitz, Rose and Speer's and Zhidkov's works.

On Randomized Data

In Bourgain's case, for the cubic NLS on $\mathbb{T}^2$, the typical element in the support of the Gibbs measure (the invariant measure) consists of **randomized data**:

$$\phi^\omega(x) = \sum \frac{g_n(\omega)}{|n|} e^{i\langle x,n\rangle} \in H^{-\varepsilon}(\mathbb{T}^2),$$

where $\{g_n(\omega)\}_n$ are i.i.d. standard (complex) (Gaussian) random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Remark

Note that if

$$\phi(x) = \sum_{n\in\mathbb{Z}^2} \frac{1}{|n|} e^{i\langle x,n\rangle},$$

then $\phi \in H^{-\epsilon}$ and also $\phi^\omega(x)$ defines almost surely in ω a function in $H^{-\epsilon}$ but **not** *in $H^s, s \geq 0$!*

In other words *randomization* does **not** improve regularity in terms of derivatives!

Randomization = Better estimates

But there is an *almost sure* improved regularity -akin to the role of Kintchine inequalities in Littlewood-Paley theory- that stems from classical results of **Rademacher, Kolmogorov**, **Paley** and **Zygmund** proving that random series on the torus enjoy better L^p bounds than deterministic ones.
For example, consider the *Rademacher Series* :

$$f(\omega) := \sum_{n=0}^{\infty} a_n \, r_n(\omega) \qquad \omega \in [0,1), \quad a_n \in \mathbb{C}$$

where

$$r_n(\omega) := \operatorname{sign} \sin(2^{n+1} \pi \, \omega)$$

We have:

- If $a_n \in \ell^2$ the sum $f(\omega)$ converges a.e. and moreover

Classical Theorem

If $a_n \in \ell^2$ then the sum $f(\omega)$ belongs to $L^p([0,1))$ for all $p \geq 2$. More precisely,

$$\left(\int_0^1 |f|^p \, d\omega \right)^{1/p} \approx_p \| \, a_n \, \|_{\ell^2}$$

Large Deviation-type Estimates

Proposition

Let $\{g_n(\omega)\}$ be a sequence of complex i.i.d. zero mean Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{c_n\} \in \ell^2$. Define

$$F(\omega) := \sum_n c_n g_n(\omega)$$

Then, there exists $C > 0$ such that for every $q \geq 2$ we have

$$\left\| \sum_n c_n g_n(\omega) \right\|_{L^q(\Omega)} \leq C\sqrt{q} \left(\sum_n |c_n|^2 \right)^{\frac{1}{2}} .$$

As a consequence from Chebyshev's inequality there exists $C > 0$ such that for every $\lambda > 0$,

$$\mathbb{P}(\{\omega \,:\, |F(\omega)| > \lambda\}) \leq \exp\left(\frac{-C\lambda^2}{\|F(\omega)\|^2_{L^2(\Omega)}} \right).$$

Large Deviation-type Estimates

More generally one uses the following, where k would represent the number of random terms in a multilinear estimate at hand,

Proposition (Large Deviation-type)

Let $d \geq 1$ and $c(n_1, \ldots, n_k) \in \mathbb{C}$. Let $\{(g_n)\}_{1 \leq n \leq d}$ as above. For $k \geq 1$ denote by $A(k, d) := \{(n_1, \ldots, n_k) \in \{1, \ldots, d\}^k,\ n_1 \leq \cdots \leq n_k\}$ and

$$F_k(\omega) = \sum_{A(k,d)} c(n_1, \ldots, n_k) g_{n_1}(\omega) \ldots g_{n_k}(\omega).$$

Then for $p \geq 2$

$$\|F_k\|_{L^p(\Omega)} \lesssim \sqrt{k+1}(p-1)^{\frac{k}{2}} \|F_k\|_{L^2(\Omega)}.$$

As a consequence from Chebyshev's inequality for every $\lambda > 0$,

$$\mathbb{P}(\{\omega : |F_k(\omega)| > \lambda\}) \leq \exp\left(\frac{-C\,\lambda^{\frac{2}{k}}}{\|F_k(\omega)\|_{L^2(\Omega)}^{\frac{2}{k}}}\right).$$

This result follows from the hyper-contractivity property of the Ornstein-Uhlenbeck semigroup by writing $G_n = H_n + iL_n$ where $\{H_1, \dots, H_d, L_1, \dots L_d\}$ are real centered independent Gaussian random variables with the same variance.

(**c.f. Tzvetkov; Thomann-Tzvetkov**)

Given $\delta > 0$, the large deviation result above with

$$\lambda = \delta^{-\frac{k}{2}}\|F_k(\omega)\|_{L^2(\Omega)}$$

says that in a set Ω_δ with $\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta}}$ we can replace $|F_k(\omega)|^2$ by $\|F_k(\omega)\|_{L^2(\Omega)}^2$.

An Example from Bourgain's Work:

Take again

$$\phi^{\omega}(x) = \sum \frac{g_n(\omega)}{|n|} e^{i\langle x,n\rangle}$$

and look at the cubic Wick ordered nonlinearity, involving its free evolution $S(t)\phi^{\omega}(x)$, and that Bourgain had to estimate in L^2:

$$\|F_3(\omega)\|_{l^2_n l^2_m},$$

where

$$F_3(\omega) = \sum_{S_{n,m}} \frac{1}{|n_1|}\frac{1}{|n_2|}\frac{1}{|n_3|} g_{n_1}(\omega)\overline{g_{n_2}}(\omega)g_{n_3}(\omega)$$

where

$$S_{n,m} = \{(n_1,n_2,n_3)\,/\,n_1 - n_2 + n_3 = n;\ n_1, n_3 \neq n_2;\ m = |n_1|^2 - |n_2|^2 + |n_3|^2\}$$

Naively we could just use C-S to estimate $\|F_3(\omega)\|^2_{l^2_n l^2_m}$ and obtain

$$\sum_{n,m}\left|\sum_{S_{n,m}}\frac{1}{|n_1|}\frac{1}{|n_2|}\frac{1}{|n_3|}g_{n_1}(\omega)\overline{g_{n_2}}(\omega)g_{n_3}(\omega)\right|^2 \lesssim \sum_{n,m}|S_{n,m}|\sum_{S_{n,m}}\frac{1}{|n_1|^2}\frac{1}{|n_2|^2}\frac{1}{|n_3|^2}$$

where $|S_{n,m}|$ is the cardinality of $S_{n,m}$ and it translates into a **loss** of derivatives.

Now in the Large Deviation Estimate, take

$$\lambda = \delta^{-1}\|F_3(\omega)\|_{L^2(\Omega)}.$$

Then in a set Ω_δ of measure $1 - e^{-\frac{1}{\delta^\alpha}}$ one has

$$\|F_3(\omega)\|^2_{l^2_n l^2_m} = \sum_{n,m}|F_3(\omega)|^2 \lesssim \delta^{-2}\sum_{n,m}\|F_3(\omega)\|^2_{L^2(\Omega)}$$

$$= \delta^{-2}\sum_{n,m}\sum_{S_{n,m}}\sum_{S'_{n,m}}\int_\Omega \frac{g_{n_1}}{|n_1|}\frac{\overline{g_{n_2}}}{|n_2|}\frac{g_{n_3}}{|n_3|}\frac{\overline{g_{n'_1}}}{|n'_1|}\frac{g_{n'_2}}{|n'_2|}\frac{\overline{g_{n'_3}}}{|n'_3|}\,d\omega$$

and by **independence** of the random variables the RHS contracts to

$$\|F_3(\omega)\|^2_{l^2_n l^2_m} \lesssim \delta^{-2}\sum_{n,m}\sum_{S_{n,m}}\frac{1}{|n_1|^2}\frac{1}{|n_2|^2}\frac{1}{|n_3|^2}$$

Randomization without invariant measure (a.s LWP)

In this vein, consider

$$\text{(IVP)} \qquad \begin{cases} u_t + P(D)u = F(u) & x \in M, \ t > 0 \\ u(x,0) = \phi(x), \end{cases}$$

with $\phi \in X^s$, a set of initial data of regularity s small.

- If M is a compact manifold of dimension d, no boundaries and with a countable basis of eigenvectors $\{h_n(x)\}$ for the Laplacian, then we randomize ϕ as
$$\phi^\omega(x) := \sum_{n \in \mathbb{Z}^d} a_n \, g_n(\omega) \, h_n(x).$$
- If $M = \mathbb{R}^d$ then we randomize ϕ as
$$\phi^\omega(x) := \sum_{n \in \mathbb{Z}^d} g_n(\omega) P_n \phi(x),$$
where P_n is a projection operator on cubes of size *one* on the frequency space.

General procedure to prove a.s LWP

- Assume v^ω is the **linear evolution** with initial datum ϕ^ω.
- Use the fact that v^ω has better L^p or multilinear estimates than ϕ *almost surely* to show that $w = u - v^\omega$ solves a **difference equation** that lives in a smoother space than X^s. Obtain for w a *deterministic* local well-posedness.

Remark (Important)

The difference equation that w solves is not back to merely being at a 'smoother' level but rather it is a hybrid equation with nonlinearity $=$
$$= \quad \textit{supercritical (but random)} + \textit{deterministic (smoother)}.$$

a.s Local to a.s Global: known mechanisms

- Invariant Gibbs or weighted Wiener measures- when available.

  The use of the invariance of the **measure** has limitations since in **higher dimensions** its support (data) lives on extremely **rough spaces** where the multilinear analysis needed to control the nonlinear terms of the equation is so far not possible. **In higher dimensions usually a radial assumption is put in place**.

- Sometimes may use energy methods (**Burq-Tzvetkov** and **Pocovnicu** for NLW; **Nahmod-Pavlovic-S.** for NS)

- Sometimes may use adaptation to this setting of Bourgain's *high-low method* (**Colliander- T. Oh**, NLS; **Bulut**, **Luehrmann-Mendelson**, NLW)

  These methods have limitations!

- Randomization techniques have now been used with or without the help of the invariant measure in several contexts:

After Bourgain's work in 94-96'; in 07-08 work by Burq-Tzvetkov (NLW, supercritical), T. Oh's (coupled KdV system, subcritical) and Tzvetkov (NLS, subcritical). Lots of work followed:

  - Schrödinger Equations: **Bourgain**, **Tzvetkov**, **Thomann**, **Thomann-Tzvetkov**, **Nahmod-Oh-Rey-Bellet-S.**, **Nahmod-Rey-Bellet-Sheffield-S.**, **Burq-Thomann-Tzevtkov**, **Y. Deng**, **Burq-Lebeau**, **Bourgain-Bulut**, **Nahmod- S**, **Poiret-Robert-Thomann**, **Bényi- Oh-Pocovnicu**.
  - KdV Equations: **Bourgain**, **T. Oh** and **Richards**.
  - NLW Equations: **Burq-Tzvetkov**, **de Suzzoni**, **Bourgain-Bulut** and **Luehrmann-Mendelson**. Also see non-squeezing for 3D cubic NLKG **Mendelson**.
  - Benjamin-Ono Equations: **Y. Deng** and **Y. Deng-Tzvetkov-Visciglia**.
  - Navier-Stokes Equations: **Nahmod-Pavlovic-S.** (infinite energy weak solutions on $\mathbb{T}^3$). Also work by **C.Deng-Cui** and **Zhang-Fang**

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To sum up:

- When deterministic statements about existence, uniqueness and stability of solutions to certain evolution equations are **not** feasible/available:

  $\rightarrow$ turn to a more probabilistic point of view

  $\rightarrow$ within reach at this time: investigate these problems from a nondeterministic viewpoint; e.g. for random data.

Situations when such a point of view is desirable include:

- supercritical regime
- when certain type of illposedness is present,
- when there still remains a gap between local and global wellposedness (subcritical regime relative to the scaling threshold),

Three Problems Solved Via Data Randomization

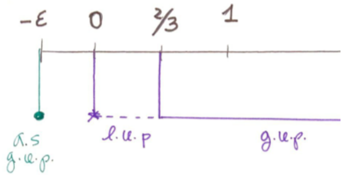
- *Result 1* : A.s. global well-posedness for 2D, cubic defocusing NLS in $H^s(\mathbb{T}^2)$, $0 < s < 1$. **(Nahmod-S.)**
- *Result 2* : Existence of large data global solutions to the 3D quintic NLS for supercritical data in $H^{1-\epsilon}(\mathbb{T}^3)$. **(Nahmod-S.)**
- *Result 3* : A.s. global well-posedness for 1D, quintic (small mass) focusing NLS in $H^s(\mathbb{T})$, $1/2 < s$. **(Nahmod-S.)**

On *Result 1*

Theorem (Nahmod-S.)

The 2D cubic defocusing NLS is a.s globally well-posed in $H^s(\mathbb{T}^2)$, $0 < s < 2/3$.

What Was Known:



- Deterministic methods: l.w.p for $s > 0$ **(Bourgain)** and g.w.p. $s > 2/3$ **(Bourgain; De Silva-Pavlovic-S.-Tzirakis)**.
- Methods exploiting data randomization and invariant Gibbs measure μ: a.s. global well-posedness in $H^{-\epsilon}$, **(Bourgain)**.

Remark: The theorem is not trivial since any $\Sigma \subset H^s, s > 0$, is such that for the Gibbs measure μ one has $\mu(\Sigma) = 0$.

Idea of the Proof

Let $\{g_n(\omega)\}$ be a sequence of complex i.i.d. zero mean Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then consider the data

$$\phi_\alpha^\omega(x) = \sum \frac{1}{|n|^\alpha} \frac{g_n(\omega)}{|n|} e^{i\langle x,n\rangle}$$

for $\alpha > 0$. Note that $\phi_\alpha^\omega \in H^s$, $0 < s < \alpha$.

- **Step 1:** Prove that a.s with these data the IVP is globally well-posed in $H^{-\epsilon}$.

  Pick $N \gg 1$ and write

  $$\phi_\alpha^\omega(x) = \sum_{|n|<N} \frac{g_n(\omega)}{|n|^{1+\alpha}} e^{in\cdot x} + \sum_{n\in\mathbb{Z}^2} a_n \frac{g_n(\omega)}{|n|} e^{in\cdot x} =: w_N(x) + \psi_1^\omega(x)$$

  where $\|w_N\|_{H^\epsilon} \leq A$ and

  $$a_n = \begin{cases} 0 & \text{if} \quad |n| < N \\ \frac{1}{|n|^\alpha} & \text{if} \quad |n| \geq N, \end{cases} \qquad |a_n| \lesssim \frac{1}{N^\alpha} \quad \text{for all} \quad n \in \mathbb{Z}^2.$$

Use Bourgain's result in $H^{-\epsilon}$ to claim that there exists a set $\tilde{\Omega} \subset \Omega$ such that $\mathbb{P}(\tilde{\Omega}) = 1$ and for any $\omega \in \tilde{\Omega}$ one has that

$$\phi^{\omega}_{\beta_N}(x) = \beta_N \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{|n|} e^{in\cdot x} + w_N(x) =: \psi^{\omega}_2(x) + w_N(x)$$

where $\beta_N = \frac{1}{N^{\alpha}}$ and $\phi^{\omega}_{\beta_N}(x)$ evolves globally to a solution v_N and

$$v_N(t) \in S(t)(\phi^{\omega}_{\beta_N}) + H^{\epsilon}.$$

Use the global solution v_N and a *perturbation argument* to prove the existence and uniqueness of the solution u for the original IVP

$$u(t) \in S(t)(\phi^{\omega}_{\alpha}) + H^{\epsilon}.$$

The **Key** point is that if we define $\zeta = v_N - u$, then ζ solves a Schrödinger equation with nonlinearity of type

$$F([v_N + S(t)w_N] + S(t)\psi^{\omega}_1) - F([v_N + S(t)w_N] + S(t)\psi^{\omega}_2 - \zeta),$$

where ψ^{ω}_1, ψ^{ω}_2 are *small* and w_N is *uniformly bounded* in N.

- **Step 2**: Recovery of regularity to claim:

$$u(t) \in S(t)(\phi_\alpha^\omega) + H^{\alpha+\epsilon}.$$

From **Step 1** we know that $u(x,t) = S(t)(\phi_\alpha^\omega) + w(x,t)$ with $\|w\|_{H^\epsilon} \leq A$. We want to upgrade the regularity of w by D^α. By the Duhamel principle we have to estimate

$$\begin{aligned} D^\alpha w &= \int_0^t S(t-t')D^\alpha[|S(t')(\phi_\alpha^\omega) + w|^2(S(t')(\phi_\alpha^\omega) + w)]\,dt' \\ &\sim \int_0^t S(t-t')[(S(t')(\phi^\omega) + D^\alpha w)(S(t')(\phi_\alpha^\omega) + w)^2]\,dt' \end{aligned}$$

For which the analysis of Bourgain in the random part $S(t')(\phi^\omega)$ and the fact that $D^\alpha w$ only appears **linearly** can be used to conclude.

On *Result 2*

We consider the energy-critical quintic nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = \lambda u|u|^4 & x \in \mathbb{T}^3 \quad \textbf{(Rational)} \\ u(0,x) = \phi(x) & \in \ H^\gamma(\mathbb{T}^3), \end{cases}$$

below $H^1(\mathbb{T}^3)$ (ie. for some $\gamma < 1$) and where $\lambda = \pm 1$

- Herr, Tzvetkov and Tataru (10') proved small data global well posedness in $H^1(\mathbb{T}^3)$.
- Ionescu and Pausader (12') proved *large data* global well posedness in $H^1(\mathbb{T}^3)$ in the defocusing case
  - Rely on large data GWP in $\mathbb{R}^3$ for the energy-critical quintic NLS (by Colliander-Keel-S-Takaoka-Tao (03')).

Our interest is first to establish a local almost sure well posedness for random data *below* $H^1(\mathbb{T}^3)$ that is in the **supercritical** regime relative to scaling, and then address g.w.p.

The Initial Data

The problem we are considering here is the analogue of the supercritical well-posedness result proved by **Bourgain** for the periodic mass critical cubic NLS in 2D; that is - a.s for data in $H^{-\epsilon}(\mathbb{T}^2)$, $\epsilon > 0$ mentioned above.

In our problem we consider data $\phi \in H^{1-\varepsilon}(\mathbb{T}^3)$ for any $\varepsilon > 0$ of the form

$$\phi(x) = \sum_{n \in \mathbb{Z}^3} \frac{1}{\langle n \rangle^{\frac{5}{2}}} e^{in \cdot x} \quad \xrightarrow{\text{randomization}} \quad \phi^\omega(x) = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle^{\frac{5}{2}}} e^{in \cdot x}$$

where $(g_n(\omega))_{n \in \mathbb{Z}^3}$ is a sequence of complex i.i.d centered Gaussian random variables, as above.

Heart of the matter

- Assume u solves our IVP, then we define $w := u - S(t)\phi^{\omega}$, where $S(t)\phi^{\omega}$ is the linear evolution of the initial profile ϕ^{ω}.

- We study the IVP for w which solves a difference equation with nonlinearity

$$\tilde{N}(w) := |w + S(t)\phi^{\omega}|^4 (w + S(t)\phi^{\omega}).$$

  We expect to prove that w belongs to H^s for some $s > 1$.

- The heart of the matter is to prove multilinear deterministic/random estimates coming from $\tilde{N}(w)$ to then be able to set up a contraction method to obtain well-posedness.

- When the NLS equation is considered, **multilinear estimates for** $\tilde{N}(w)$ can be carried out only after having removed certain *"double frequencies"* involved in the nonlinear part of the equation. In the **cubic** case a Wick ordering of the Hamiltonian was used (see **Bourgain (96'), Colliander-Oh (12')**).

Analogies and Difficulties for Local Well-posedness

There are four major complications in the work that we present here compared to the work of Bourgain:

- a quintic nonlinearity increases quite substantially the different cases that needs to be analyzed,
- the counting lemmata in a $3D$ integer lattice are much less favorable than in a $2D$ lattice,
- the Wick ordering is not sufficient to remove certain bad *resonant* frequencies.
- We work on [HTT]'s atomic function spaces $X^s, \ Y^s$ whose norms are not invariant if one replaces the Fourier transform with its absolute value.

Main Results:

Theorem (Nahmod-S.)

Let ϕ^ω as above. Then there exists $s > 1$ and and there exists $0 < \delta_0 \ll 1$ and $r = r(s) > 0$ s.t. for any $\delta < \delta_0$, there exists Ω_δ with

$$\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta^r}},$$

and for each $\omega \in \Omega_\delta$ there exists a unique solution u of the quintic NLS in the space

$$S(t)\phi^\omega + X^s([0,\delta))_d,$$

with initial condition ϕ^ω.

On Global Solutions

Extending these solutions globally in time is hard since if there were an invariant Gibbs measure it would be supported in $H^{-1/2-\epsilon}$! Other possible routes could be:

- Low-High method of Bourgain and randomization. (See previous work of **Colliander-Oh** and **Luehrmann-Mendelson**). This though is not implementable at the moment because one would need to use the global result in H^1 by Ionescu-Pausader where the bounds for the "Strichartz norm" of the solution is super-exponential with respect to the energy.
- The recent *conditional* argument of **Bényi- Oh- Pocovnicu** for the NLS.

  See also the recent results of **Pocovnicu** and **Oh-Pocovnicu** similar to the one above but for NLW. Here they are able to remove the *conditional assumption* by using a "probabilistic" energy bound on the difference equation.

On Global Solutions

If we are interested on claiming that there are "some" large data evolving to solutions for large times, then this can be done. Again assume

$$\phi^\omega = \sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^{\frac{5}{2}}} e^{in\cdot x}.$$

We have the following theorem:

Theorem (Nahmod-S.)

Let $s > 1$ and ϕ^ω as above. Fix a large interval of time $[0, T]$. Then there exists $0 < \delta \sim T^{-\frac{1}{4}}$ and there exists Ω_δ with

$$\mathbb{P}(\Omega_\delta^c) < e^{-\delta}$$

and for each $\omega \in \Omega_\delta$ there exists a unique solution u of the quintic NLS in the space

$$S(t)\phi^\omega + X^s([0, T))_d,$$

with initial condition ϕ^ω.

Remark

- *This is a large data result.*
- *As $T \to \infty$ the size of the set of initial data giving rise to solutions on the whole interval $[0, T]$ shrinks to zero.*

Idea of the proof: It is a combination of an iterated continuity argument and the fact that the random term can be made small via Large Deviation Estimates.

Remark

Krieger-Schlag *considered the septic NLW in $\mathbb{R}^{3+1}$ and proved the existence of a class of global smooth solutions with infinite critical norm $\dot{H}^{7/6} \times \dot{H}^{1/6}$. This is a constructive purely deterministic approach.*

On *Result 3*

Theorem (Nahmod-S.)

The focusing 1D quintic NLS, with small mass, is a.s globally well-posed in $H^s(\mathbb{T}^1)$, $1/2 < s$.

What Was Known:

- Deterministic methods, focusing and defocusing: l.w.p for $s > 0$, **(Bourgain)**.
- For defocusing g.w.p for $s > 4/9$, **(Bourgain), (De Silva-Pavlovic-S.-Tzirakis)**.
- Methods exploiting data randomization: a.s. global well-posedness in $H^{1/2-\epsilon}$ for defocusing and focusing when mass small, **(Bourgain)**.

Remark: Also in this case the theorem is not trivial since any $\Sigma \subset H^s, s > 1/2$, is such that for the Gibbs measure μ one has $\mu(\Sigma) = 0$.

Idea of the Proof

Let $\{g_n(\omega)\}$ be a sequence of complex i.i.d. zero mean Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then consider the data

$$\phi_\alpha^\omega(x) = \sum \frac{1}{|n|^\alpha} \frac{g_n(\omega)}{|n|} e^{i\langle x,n\rangle}$$

for $\alpha > 0$. Note that $\phi_\alpha^\omega \in H^s$, $0 < s < 1/2 + \alpha$.

- **Step 1:** Prove that a.s with these data the IVP is globally well-posed in $H^{1/2-\epsilon}$. In particular show that the solution u can be written as

$$u(x,t) = S(t)\phi_\alpha^\omega(x) + w(x,t), \quad \|w(t)\|_{H^{1/2+\epsilon}} < A, \quad \forall t.$$

Remark

This step doesn't follow from Bourgain's proof since there he uses the deterministic l.w.p result available for $s > 0$. In order to obtain this step one has to repeat the argument for the quintic IVP, (gauge transformation etc). Here the analysis is simpler since the counting lemmata are trivial.

- **Step 2:** Recovery of regularity.