Wave Turbulence Closures:

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Hamiltonian Systems: $\dot{p}_k = \frac{\partial H}{\partial q_k}$, $\dot{q}_k = -\frac{\partial H}{\partial p_k}$, $k \in \mathbb{N}$.

$$p_k, q_k(t) \in \mathbb{R}^d, \quad t \in \mathbb{R}, \quad H(\vec{p}, \vec{q}) = \text{Hamiltonian}$$

1. **ODE example:** Ideal gas of N-particles in $D \subseteq \mathbb{R}^3$

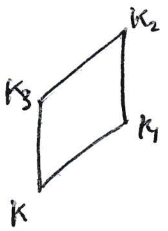
$(q_k(t), p_k(t)) \to$ pos. & momentum, $H = \frac{1}{2} \sum_{k=1}^{N} |p_k|^2$

2. Cubic (NLS) eqn $\begin{cases} (i\partial_t + \Delta) u = \varepsilon^2 |u|^2 u \\ u(0,x) = u_0 \end{cases}$

$x \in \mathbb{T}_L^d = [0,L]^d \circlearrowleft$ $\quad \boxed{d=2} \quad \mathbb{Z}_L^2 = \mathbb{Z}^2 / L^2$

$$u(t) = \frac{1}{L^2} \sum_{k \in \mathbb{Z}_L^d} \underbrace{\hat{u}(k,t)}_{= e^{-i|k|^2 t} a_k} e^{2\pi i k \cdot x}$$

$$(\text{NLS}) \begin{cases} i\partial_t a_k = \frac{\varepsilon^2}{L^4} \sum_{k_1 - k_2 + k_3 = k} a_{k_1}(t) \overline{a_{k_2}}(t) a_{k_3}(t) e^{i\Omega t} \\ \Omega = |k_1|^2 - |k_2|^2 + |k_3|^2 - |k_4|^2 \end{cases}$$



$\Omega = 0 \implies$ resonant interaction

$$(\text{RNLS}) \quad i\partial_t r_k = \frac{\varepsilon^2}{L^4} \sum_{R(k)} r_{k_1} \overline{r_{k_2}} r_{k_3}.$$

$R(k) \hookrightarrow S(k) \cap \{\Omega = 0\}$

<u>Q</u>: Long-time behavior ⇝ energy distribution

How does the "energy" of the system get transferred & redistributed among the available "d.o.f."

<u>For NLS</u> "energy" ⟶ Mass $\sum' |\hat{u}(k)|^2$

K.E. $\sum' |k|^2 |\hat{u}(k)|^2$

"d.o.f" ⟶ Fourier modes

* Dynamical systems approach: ⇝ trajectories/orbits

* Statistical Mechanics Approach: ⇝ steady states & invariant measures.

Equilibrium stat. mech steady states of isolated system

Non-equilibrium (e.g. steady states of systems w/ sources/sinks)

* Non-equilibrium stat. mech ⇝ wave turbulence theory

* <u>Wave turbulence (formal level)</u>

⇝ effective eqn for $|a_k|^2$

$a_k(0) \longrightarrow$ random variable

$n^L(k) = \mathbb{E}|a_k|^2$

$$\partial_t n^L(k) = \partial_t \mathbb{E}|a_k|^2 = 2\,\mathrm{Im}\,\mathbb{E}(i\partial_t a_k)\overline{a_k}$$

$$= 2\sum_{S(k)}' \mathrm{Im}\,\mathbb{E}(a_{k_1}\overline{a_{k_2}}a_{k_3}\overline{a_k})\, e^{i\Omega t}$$

$\partial_t \mathbb{E}(a_{k_1} \cdots a_{k_N}) \rightsquigarrow$ infinite Heirarchy for

$\mathbb{E}\, a_{k_1} \overline{a_{k_2}} \cdots a_{k_N}$

Closure Problem: Get a closed effective equation for $n^L(k,t)$?

① Independence / ergodicity assumptions (NOT justified).

② $(\varepsilon \to 0)$ Weak non-linearity limit

③ $(L \to \infty)$ Large box limit.

$\lim_{\substack{\varepsilon \to 0 \\ L \to \infty}} n^L(k) \longrightarrow n(k)$ satisfying

$$\partial_t n = \iiint \delta(k_1 - k_2 + k_3 - k)\, \delta(\Omega = 0)\, n_1 n_2 n_3 n \left(\frac{1}{n_1} - \frac{1}{n_2} + \frac{1}{n_3} - \frac{1}{n}\right) dk_1 dk_2 dk_3$$

(W.K.E.) $\qquad n_j = n(k_j)$.

Stationary Solution $\longleftrightarrow$ spectra

① Equilibrium spectra (equipartition) $n(k) = 1$, $n(k) = |k|^{-2}$

② Non-equilibrium spectra (KZ spectra)

- Direct cascade: $n(k) = |k|^{\frac{2}{3} - d}$
- Inverse cascade: $n(k) = |k|^{-d}$, Escobedo-Valasquez

Problem: Can we make this derivation rigorous?

Q: Can we make this derivation rigorous?

Realistic Problem: Can we find a randomization that allows justification of (WKE)?

* Rigorous derivation of (WKE):

$\tau > 0$

$$\begin{cases} (i\partial_t u + \Delta u) = \varepsilon^2 \langle D\rangle^{-\beta} |\langle D\rangle^{-\beta}|^2 \langle D\rangle^{-\beta} u & m\tau \le t \le (m+1)\tau \\ \hat{u}(m\tau, k) = \hat{u}(m\tau-, k)\, e^{i\theta_{m,k}}, & \hat{u}(0,k) = \hat{g}(k). \end{cases}$$

Becomes
$$i\partial_t a_k = \frac{\varepsilon^2}{L^4} \sum_{k_1-k_2+k_3=k}' T_{k_1k_2k_3k}\, a_{k_1}(t)\, \overline{a_{k_2}}(t)\, a_{k_3}(t)\, e^{i\Omega t}$$
$$m\tau \le t \le (m+1)\tau$$

where $T_{k_1k_2k_3k} = \prod_{j=0}^{3} \langle k_j\rangle^{-\beta}$

$$a_k(m\tau) = a_k(m\tau-)\, e^{i\theta_{m,k}}$$

$\{\theta_{m,k}\}_{m\in\mathbb{N},\, k\in\mathbb{Z}^2_L} \longrightarrow$ unif. dist. on $[0, 2\pi]$.

Aim: Effective equation for $n^L(k) = \mathbb{E}|a_k|^2$

$$\boxed{\begin{array}{l} \textbf{Thm.}\ \text{In the limits } \varepsilon \to 0,\ \tau \to 0,\ L \to 0 \\[6pt] \qquad \left\| n^L(k) - n\!\left(k, \frac{t}{T}\right) \right\|_{\ell^\infty} \longrightarrow 0, \\[6pt] \qquad n \text{ solves (WKE) with } n_0 = |g_0|^{2k}. \end{array}}$$

$|a_k|^2(t) \longrightarrow$ piecewise smooth & cont.

Step 1: $(\varepsilon \to 0$ limit$) \rightsquigarrow$ normal forms

$$\partial_t a_k = \sum_{R(k)}' a_{k_1} \overline{a_{k_2}} a_{k_3}$$

$S(k) \cap \{\Omega = 0\}$.

Step 2: Hierarchy for $A^N_{\tau,L} = \mathbb{E}\, |a_{k_1}|^2 \cdots |a_{k_N}|^2$

$$\partial_t A^1_{\tau,L} = \partial_t \mathbb{E}|a_k|^2$$

$$= 2\,\mathrm{Im} \sum_{R(k)}' \mathbb{E}\left(a_{k_1} \overline{a_{k_2}} a_{k_3} \overline{a_k}\right)$$

at $t = m\tau \longrightarrow \mathbb{E}\, e^{i(\theta_{m,k_1} - \theta_{m,k_2} + \theta_{m,k_3} - \theta_{m,k})} = 0$

unless $\{k_1, k_3\} = \{k_2, k\}$

$$\partial_t A^1_{\tau,L}(m\tau) = 0$$

$$\partial_t^2 A^1_{\tau,L}(m\tau) = C(\varepsilon,L,\tau) \sum_{(k_1,k_2,k_3)\in R(k)}' \mathbb{E}\,|a_{k_2}|^2 |a_{k_3}|^2 |a_{k_1}|^2$$

$$- \mathbb{E}|a_{k_1}|^2 |a_{k_3}|^2 |a_k|^2 + \mathbb{E}\,|a_{k_1}|^2 |a_{k_3}|^2 |a_k|^2$$

$$- \mathbb{E}\,|a_{k_1}|^2 |a_{k_2}|^2 |a_{k_3}|^2 + \text{h.o.t.}$$

$$\partial_t^3 A^1_{\tau,L}(m\tau) = 0$$

$$\partial_t^4 A^1_{\tau,L}(s) \lesssim C(\varepsilon,L,\tau) \quad \forall\ m\tau < s < (m+1)\tau$$

Similarly for $A^N_{\tau,L}$

Step 3 ($\tau \to 0$) $A^N_{\tau,L} \overset{\tau\to 0}{\approx} A^N_L(t, k_1, \ldots, k_N)$

$$\partial_t A^N_L = \sum_{R(k)}' F^N(A^{N+2}_L)$$

Step 4 ($L \to \infty$)

$$\frac{1}{Z(L)} \sum_{R(k)} F(k_1,k_2,k_3) \xrightarrow{L\to\infty} \iiint \delta(S=0)\,\delta(\Omega=0)\, F(k_1,k_2,k_3)\, d\vec{k}$$

$\hookrightarrow \frac{\zeta(2)}{2} L^2 \log L$

$A^N_L \approx A^N(t; k_1, \ldots, k_N)$

$\hookrightarrow$ satisfies (WKH)

Caricature problem: For $\sigma > 0$,

f_σ is cont. & p.s.

$\partial_s f_\sigma(m\sigma) = 0$

$\partial_s^2 f_\sigma(m\sigma) = \frac{2}{\sigma} F(m\sigma)$

$\partial_s^3 f_\sigma(m\sigma) = 0$

$\partial_s^4 f_\sigma(m\sigma) = O(\sigma^2)$ $\forall$ $m\sigma < s < (m+1)\sigma$

$f_\sigma \xrightarrow{\sigma\to 0} f$, $\begin{cases} \partial_s f = F \\ f(0) = f_\sigma(0) \end{cases}$.

Step 5 (uniqueness) If $n(k)$ satisfies (WKE)

$$A^N(t, k_1, \ldots, k_N) = \prod_{j=1}^{N} n(t, k_j)$$

$\hookrightarrow$ solves (WKH)

$$n^L(t,k) = A^1_{\tau,L}(k) \underset{\tau\to 0}{\approx} A^1_L(k) \underset{L\to\infty}{\approx} A^1(k) = n(k)$$

$\downarrow$ WKE $\qquad \downarrow$ WKE

Q: Time interval? A: $T^* = \dfrac{L^6}{\varepsilon^4 \tau \cdot \log L}$
