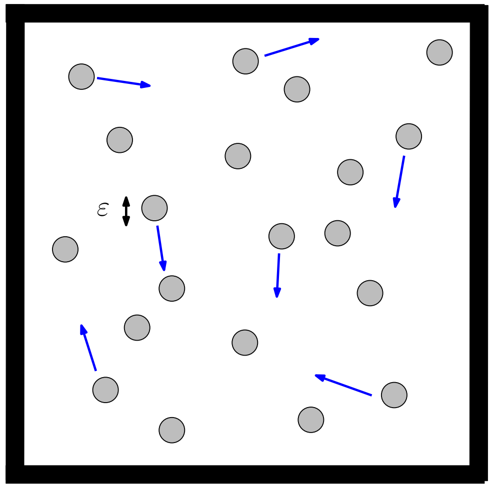
From particles to linear hydrodynamic equations

Thierry Bodineau

Joint works with Isabelle Gallagher, Laure Saint-Raymond

Outline.

- Linear Boltzmann equation & Brownian motion
- Lanford’s strategy & pruning procedure
- Coupling with the Boltzmann hierarchy



Microscopic scale

N particles of size ε

Newtonian dynamics

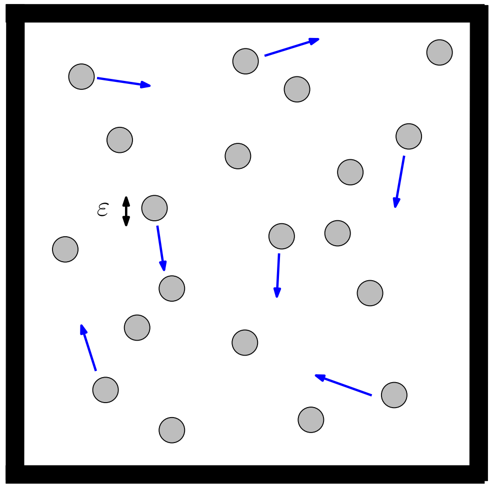
$$N\varepsilon^d \ll 1$$

$$N\varepsilon^{d-1} \gg 1$$

Macroscopic scale

Fluid equations of hydrodynamics

(Euler, Navier-Stokes)



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Stochastic perturbations:

Olla,Varadhan,Yau

Quastel, Yau ....

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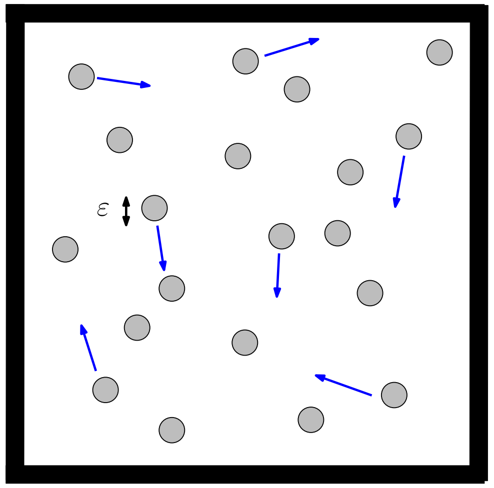


Microscopic scale

N particles of size ε

Newtonian dynamics

Low density limit

$$N\varepsilon^{d} \ll 1$$

$$N\varepsilon^{d-1} \gg 1$$

Mesoscopic scale

Boltzmann equation

Stochastic perturbations:

Olla,Varadhan,Yau

Quastel, Yau ....

Fast relaxation limit:

Bardos,Golse,Levermore

Golse,Saint-Raymond ...

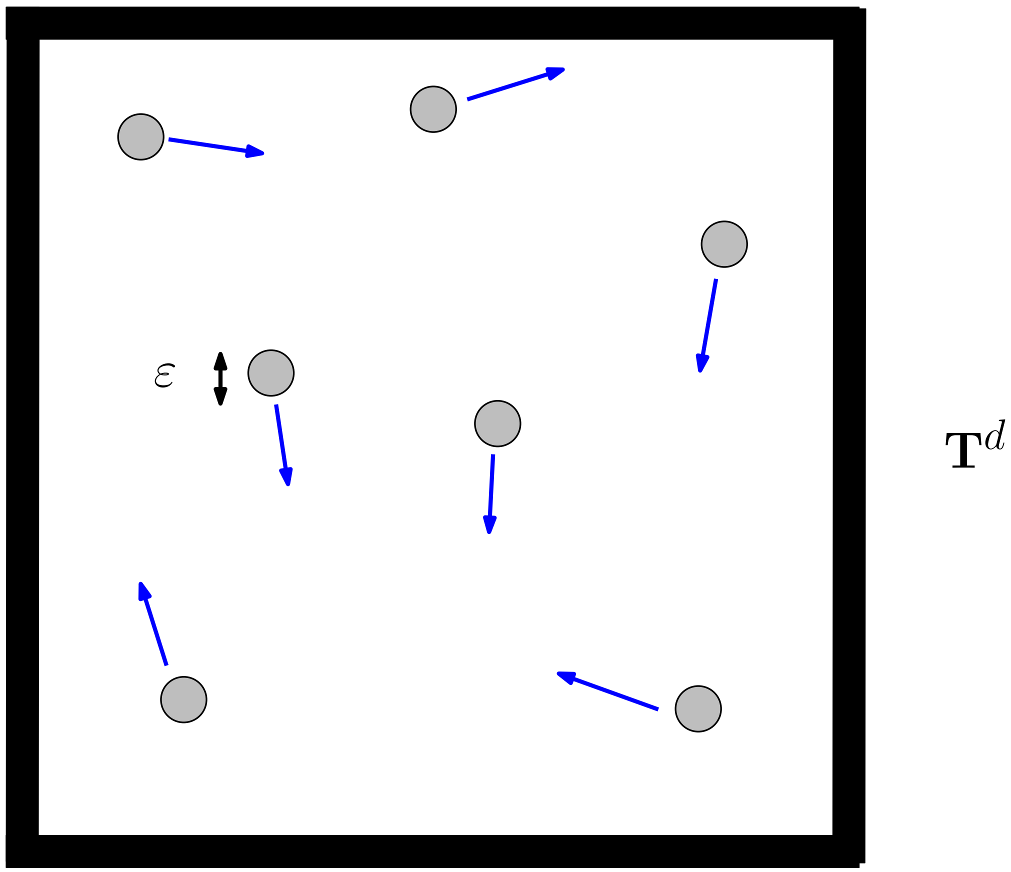
Macroscopic scale

Fluid equations of hydrodynamics

(Euler, Navier-Stokes)

Diluted Gas of hard spheres

Gas of N hard spheres with deterministic Newtonian dynamics (elastic collisions).



Dimension : $d \geq 2$

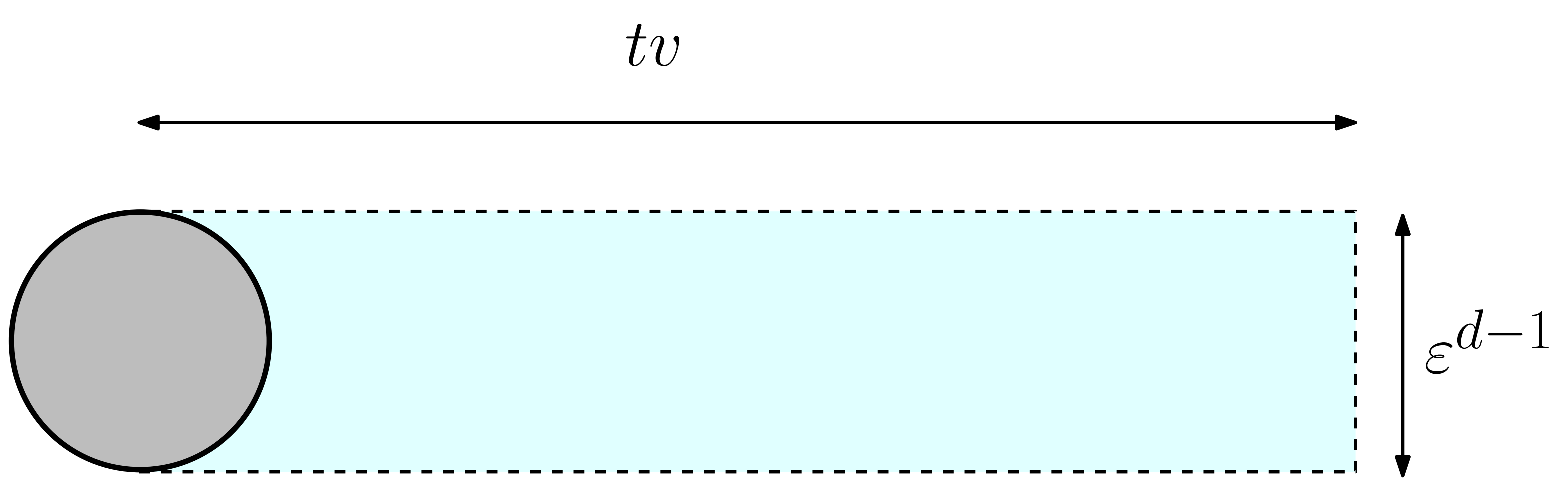
Periodic domain: $\mathbf{T}^d = [0,1]^d$

Sphere radius = ε

Boltzmann-Grad scaling

$$N\varepsilon^{d-1} = \alpha$$

Boltzmann-Grad scaling



- Volume covered by a particle $= tv\varepsilon^{d-1}$
- On average N particles per unit volume

On average, a particle has α collisions per unit of time

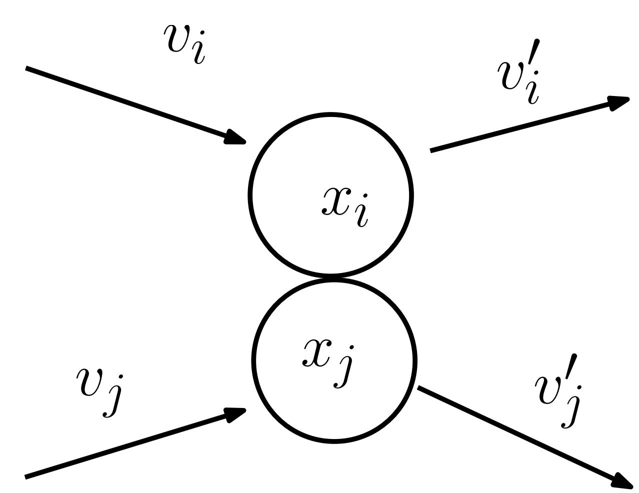
$$N \times \varepsilon^{d-1} \equiv \alpha$$

Hard Sphere dynamics

Gas of N hard spheres : $Z_N = \{(x_i(t), v_i(t)\}_{i \leq N}$

$$\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = 0 \quad \text{as long as } \ |x_i(t) - x_j(t)| > \varepsilon \, ,$$

and elastic collisions if $|x_i(t) - x_j(t)| = \varepsilon$



$$\begin{cases} v'_i + v'_j = v_i + v_j \\ |v'_j|^2 + |v'_j|^2 = |v_i|^2 + |v_j|^2 \end{cases}$$

Hard Sphere dynamics

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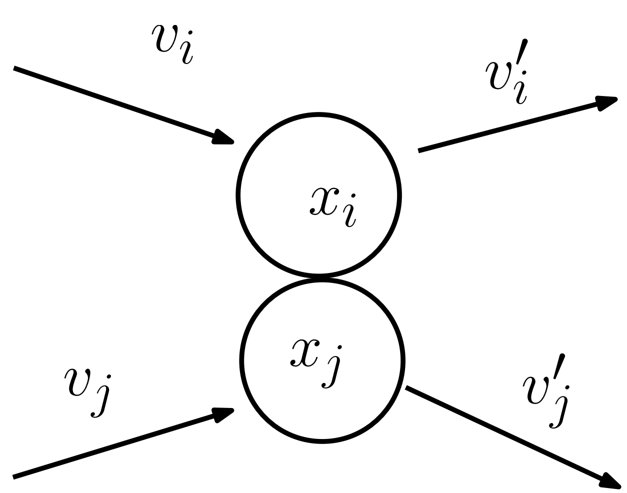

$$\begin{cases} v_i' + v_j' = v_i + v_j \\ |v_j'|^2 + |v_j'|^2 = |v_i|^2 + |v_j|^2 \end{cases}$$

Liouville equation for the particle density $f_N(t, Z_N)$

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N = 0$$

in the phase space

$$\mathcal{D}_\varepsilon^N := \{Z_N \in \mathbf{T}^{dN} \times \mathbb{R}^{dN} \,/\, \forall i \neq j\,, \quad |x_i - x_j| > \varepsilon\}$$

with specular reflection on the boundary $\partial\,\mathcal{D}_\varepsilon^N$.

Initial Data

Equilibrium distribution

$$M_{N,\beta}(Z_N) = \frac{1}{\mathcal{Z}_{N,\beta}} \exp\Big(-\frac{\beta}{2}\sum_{i=1}^{N}|v_i|^2\Big) \prod_{i\neq j} 1_{|x_i-x_j|>\varepsilon}$$

Initial data :

$$f^0_{N,\beta}(Z_N) = \left(\prod_{i=1}^{N} f^0(z_i)\right) M_{N,\beta}(Z_N)$$

Density of a particle at time t :

$$f_N^{(1)}(t, z_1) = \int dz_2 \dots dz_N \, f_N(t, z_1, z_2, \dots, z_N)$$

Question. Convergence $f_N^{(1)}(t, z_1) \xrightarrow[N\varepsilon^{d-1}=\alpha]{N\to\infty}{}^{?} f(t, z_1)$

Boltzmann equation

Theorem.

For chaotic initial data $f_N^0(Z_N) \simeq \prod_{i=1}^N f^0(z_i)$ the density of the particle system converges up to a time t >0 to the solution of the Boltzmann equation when $N \to \infty$, $N\varepsilon^{d-1} = \alpha$

$$\partial_t f + v \cdot \nabla_x f$$
$$= \alpha \iint_{\mathbf{S}^{d-1}\times\mathbb{R}^d} [f(v')f(v_1') - f(v)f(v_1)] \left((v - v_1)\cdot\nu\right)_+ dv_1 d\nu$$

with $v' = v + \nu \cdot (v_1 - v)\,\nu\,, \quad v_1' = v_1 - \nu\cdot(v_1 - v)\,\nu$

[Lanford], [King], [Alexander], [Uchiyama], [Cercignani, Illner, Pulvirenti], [Simonella], *[Gallagher, Saint-Raymond, Texier]*, [Pulvirenti, Saffirio, Simonella]

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$$\partial_t f + v \cdot \nabla_x f = \alpha \iint_{\mathbf{S}^{d-1} \times \mathbb{R}^d} [f(v')f(v'_1) - f(v)f(v_1)] \big((v - v_1) \cdot \nu\big)_+ dv_1 d\nu$$

with $v' = v + \nu \cdot (v_1 - v)\, \nu \,, \quad v'_1 = v_1 - \nu \cdot (v_1 - v)\, \nu$

Lanford's strategy leads to a short time convergence which depends on f^0. The convergence time remains short even if initially the system starts from equilibrium !!!

Large time asymptotics

Equilibrium distribution

$$M_{N,\beta}(Z_N) = \frac{1}{\mathcal{Z}_{N,\beta}} \exp\Big(-\frac{\beta}{2}\sum_{i=1}^{N}|v_i|^2\Big) \prod_{i\neq j} 1_{|x_i-x_j|>\varepsilon}$$

Initial data for Lanford's theorem

$$f^0_{N,\beta}(Z_N) = \Big(\prod_{i=1}^{N} f^0(z_i)\Big) M_{N,\beta}(Z_N) \longleftarrow \simeq \exp(N)$$

Perturbation of the equilibrium distribution :

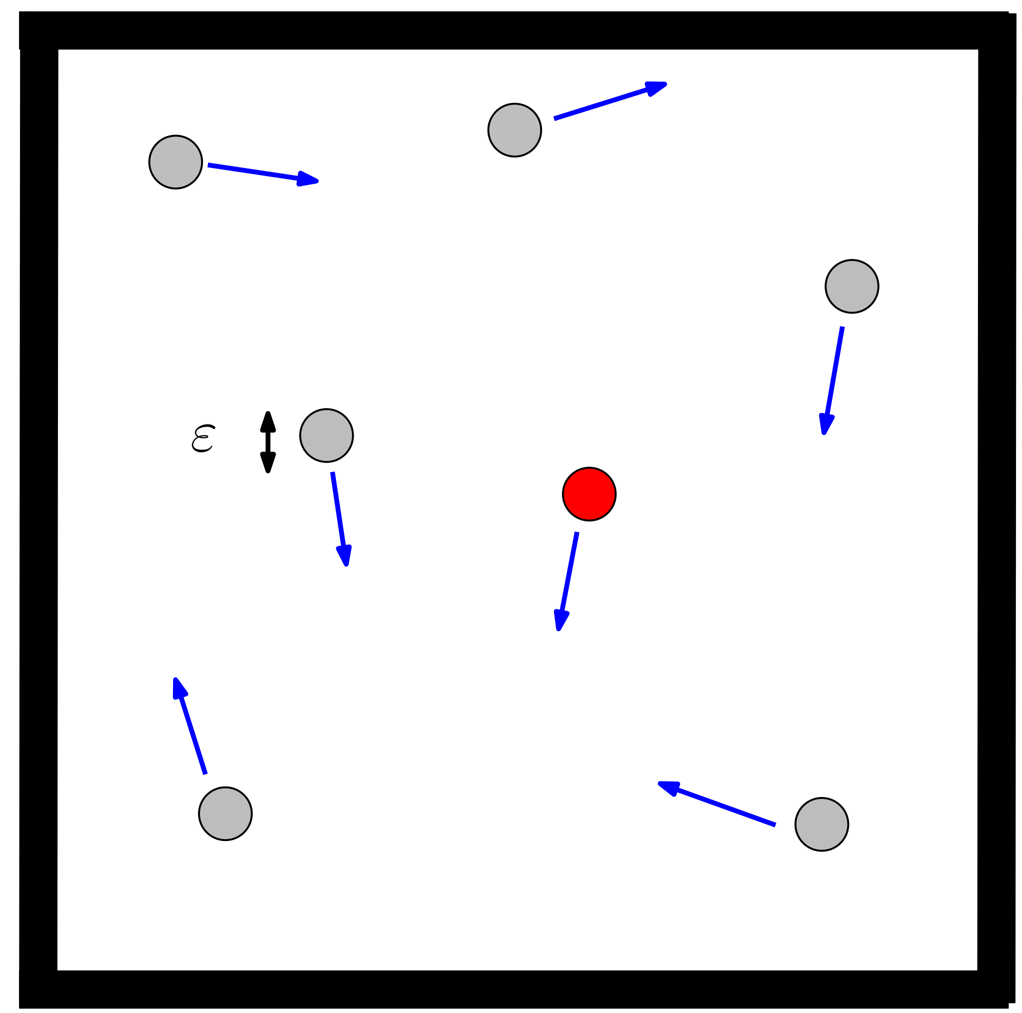
- *Linear* Boltzmann equation: perturbation of a *tagged* particle
- *Linearized* Boltzmann equation

The tagged particle

Gas of N hard spheres with deterministic Newtonian dynamics (elastic collisions).

Initial data at equilibrium and a tagged particle (x_1, v_1)

Questions.



In the Boltzmann-Grad scaling

$$N \times \varepsilon^{d-1} \equiv \alpha \text{ and } N \to \infty$$

1. Distribution of $(x_1(t), v_1(t))$

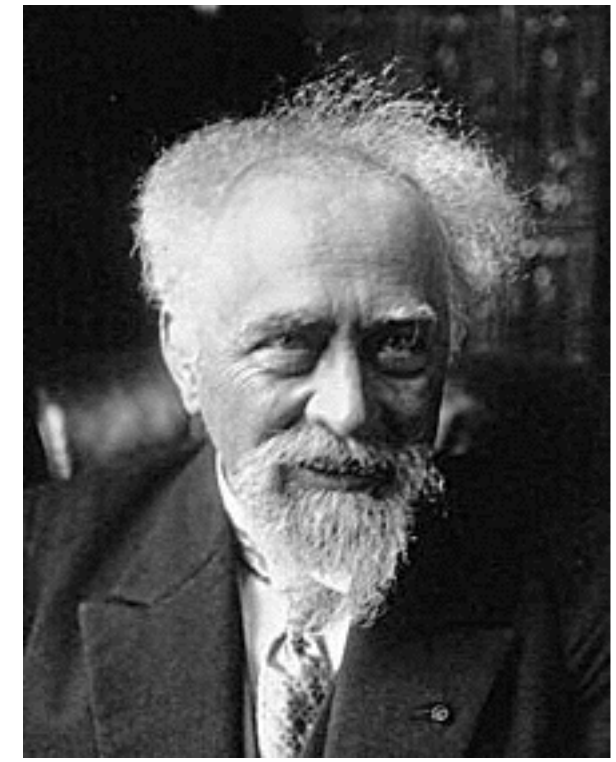
2. Position of the tagged particle

$$x_1(\alpha t) \text{ when } \alpha \to \infty$$

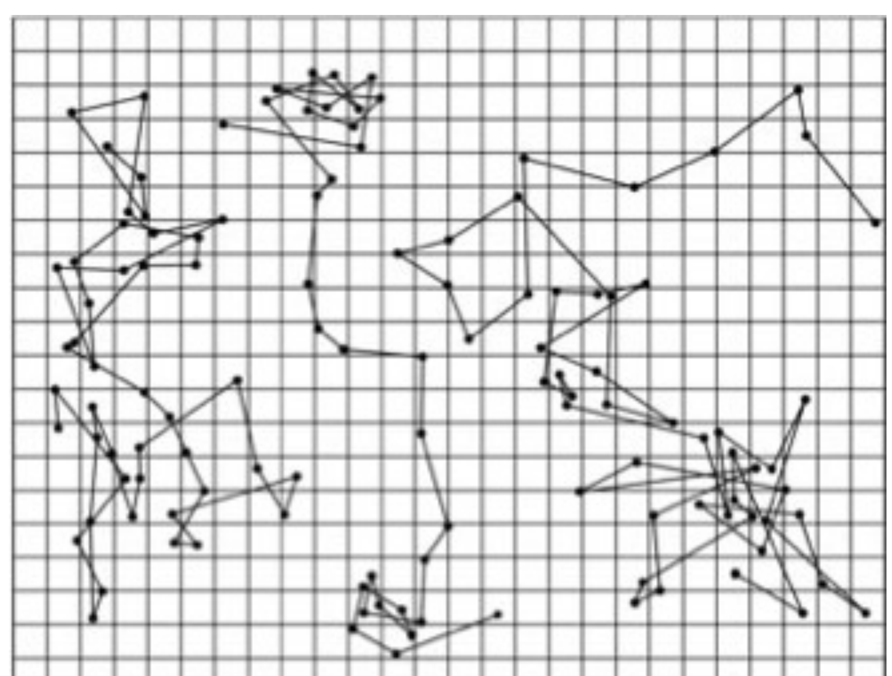
Tagged particle & Brownian motian

R. Brown
(1828)

A. Einstein
(1905)



J. Perrin
(1909)



J. Perrin
Les atomes

The tagged particle

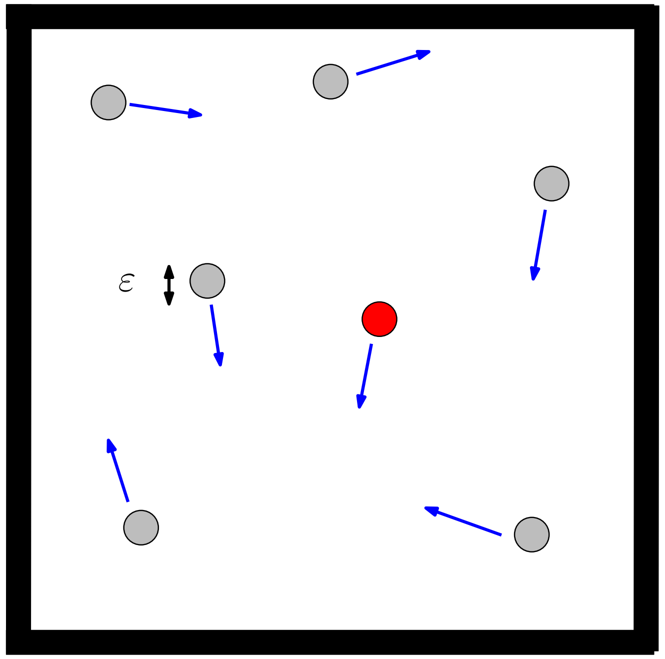
Equilibrium distribution

$$M_{N,\beta}(Z_N) = \frac{1}{\mathcal{Z}_{N,\beta}} \exp\Big(-\frac{\beta}{2}\sum_{i=1}^{N}|v_i|^2\Big) \prod_{i\neq j} 1_{|x_i - x_j| > \varepsilon}$$

Particle $Z_1 = (x_1, v_1)$ is tagged. Initial distribution :

$$f_N^0(Z_N) = M_{N,\beta}(Z_N)\,\rho^0(x_1)$$

Uniform bound: $\rho^0(x_1) \le \mu$

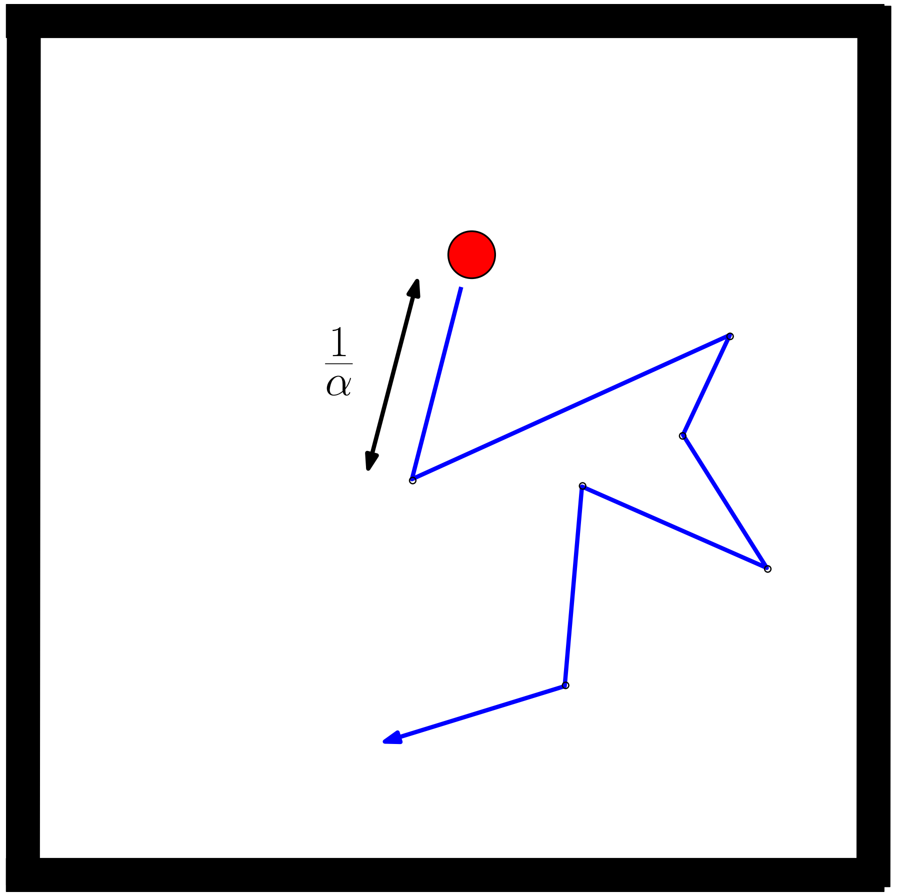


Notation: Marginals

$t \ge 0, \forall s \ge 1,$ $$f_N^{(s)}(t, Z_s) = \iint f_N(t, Z_N)\, dz_{s+1} \dots dz_N$$

Tagged particle distribution $f_N^{(1)}(t, (x_1, v_1))$

Limiting stochastic process



single particle dynamics

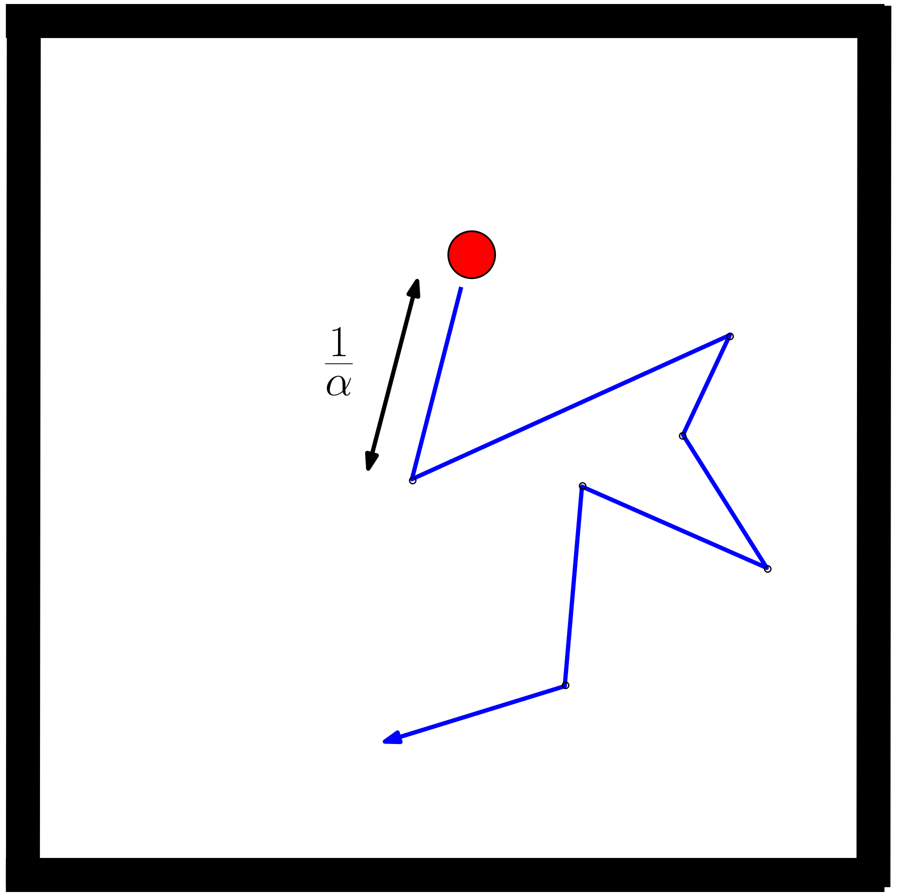
Position : $x(t) = \int_0^t v(u)du$

Markov process on the velocities $\{v(t)\}_{t\geq 0}$ with generator αL

$$Lg(v) := \iint [g(v) - g(v')] \big((v - v_1) \cdot \nu\big)_+ M_\beta(v_1)\, dv_1 d\nu$$

$$v' = v + (\nu \cdot (v_1 - v))\, \nu, \quad v_1' = v_1 - (\nu \cdot (v_1 - v))\, \nu$$

Limiting stochastic process



single particle dynamics

Position : $x(t) = \int_0^t v(u)du$

Markov process on the velocities

$\{v(t)\}_{t \geq 0}$ with generator αL

Particle distribution $M_\beta(v)\varphi_\alpha(x, v, t)$ follows the

Linear Boltzmann equation $\partial_t \varphi + v \cdot \nabla_x \varphi = -\alpha L \varphi$

Probabilist approaches :

Tanaka, Sznitman, Méléard, Fournier, Rezakhanlou ...

[van Beijeren, Lanford, Lebowitz, Spohn]

N particle system

$f_N^{(1)}(x_1, v_1, t)$

$\alpha = N\varepsilon^{d-1}$

$N \to \infty$

$t > 0$

Linear Boltzmann equation

$\varphi_\alpha(x_1, v_1, t)\, M_\beta(v_1)$

[van Beijeren, Lanford, Lebowitz, Spohn]

N particle system

$f_N^{(1)}(x_1, v_1, t)$

$\alpha = N\varepsilon^{d-1}$

$N \to \infty$

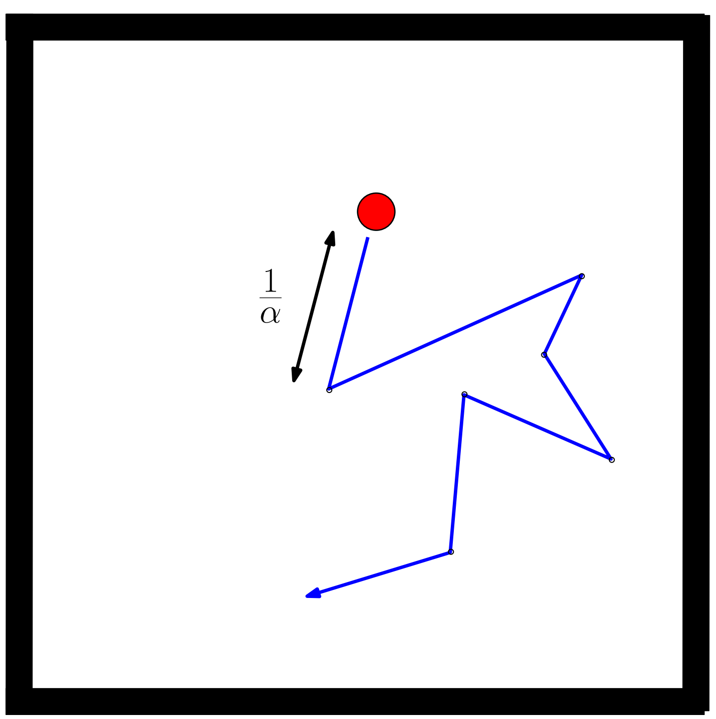
$t > 0$

Linear Boltzmann equation

$\varphi_\alpha(x_1, v_1, t)\, M_\beta(v_1)$

$t = \alpha\, \tau$

$\alpha \to \infty$

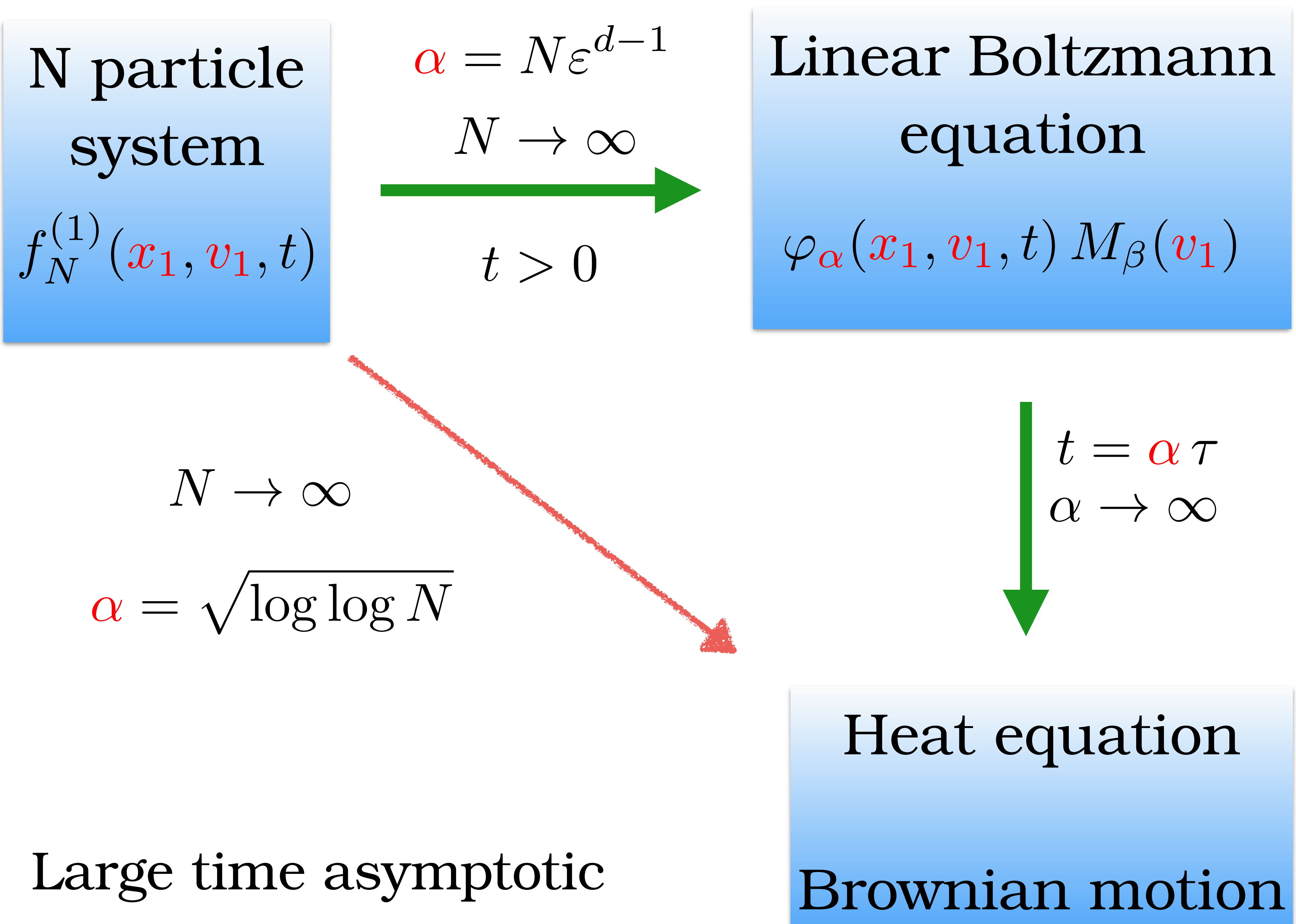


Large time asymptotic

Heat equation

Brownian motion

[van Beijeren, Lanford, Lebowitz, Spohn]



Convergence to the Brownian motion

Rescaled position of the tagged particle

$$\chi(\tau) = x_1(\alpha\tau) \quad \text{with} \quad \alpha = \sqrt{\log\log N}$$

Initial data $\quad f_N^0(Z_N) = M_{N,\beta}(Z_N)\,\rho^0(x_1)$

Theorem [B.,Gallagher, Saint-Raymond]

χ converges weakly to a brownian motion with variance κ_β

The distribution of the tagged particle $f_N^{(1)}(x_1, v_1, \alpha\tau)$

converges as $N \to \infty$ to $M_\beta(v_1)\,\rho(x_1,\tau)$

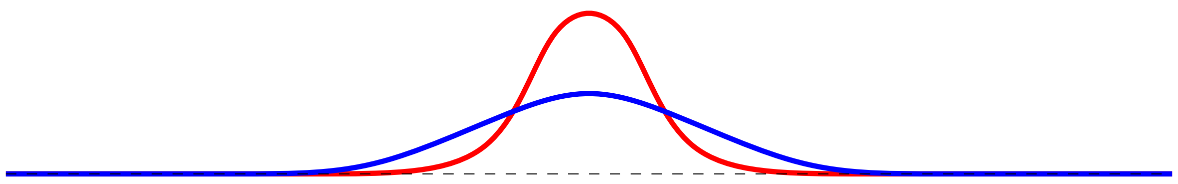
$$\partial_\tau \rho = \kappa_\beta \Delta_x \rho \quad \text{on } \mathbb{R}^+ \times [0,1]^d, \qquad \rho_{|\,\tau=0} = \rho^0$$

Quantum brownian motion: [Erdös, Salmhofer, Yau]

Lorentz gas: [Bunimovich,Sinai], [Basile,Nota,Pezzotti,Pulvirenti]

Linearized Boltzmann equation

Response to a small perturbation

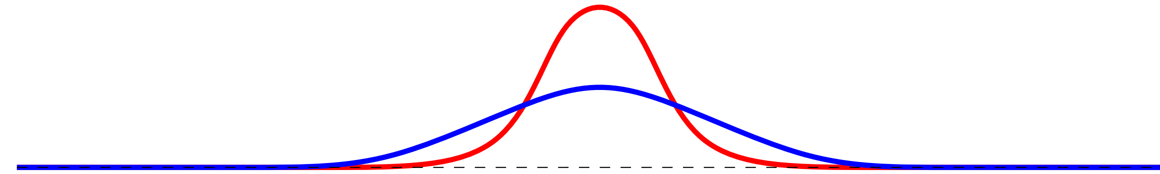


$$(\partial_t + v \cdot \nabla_x) g = -\alpha \mathcal{L} g,$$

$$\mathcal{L} g(v) := \int M_\beta(v_1) \Big(g(v) + g(v_1) - g(v') - g(v_1') \Big) \Big((v_1 - v) \cdot \nu \Big)_+ d\nu dv_1$$

Linearized Boltzmann equation

Response to a small perturbation



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Background

Linear Boltzmann equation

$$Lg(v) := \int [g(v) - g(v')] \left((v - v_1) \cdot \nu \right)_+ M_\beta(v_1) \, dv_1 d\nu$$

Derivation of the linear Boltzmann equation

Step 1. Control of the collision operators

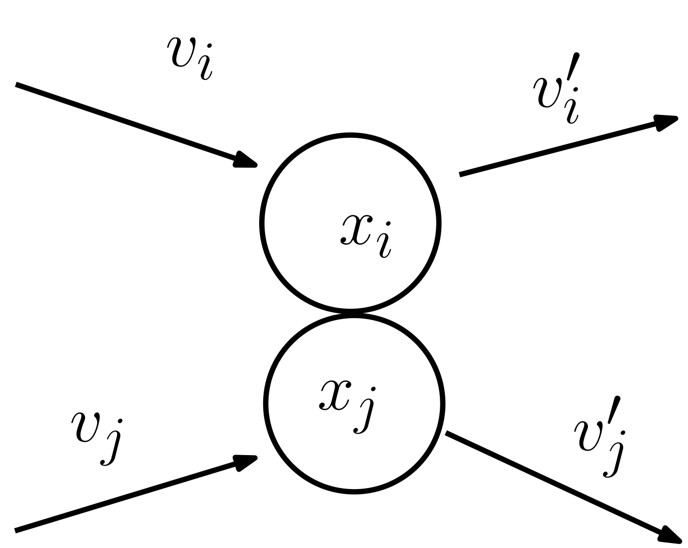
BBGKY hierarchy for the marginals

Evolution of the first marginal

$$(\partial_t + v_1 \cdot \nabla_{x_1}) f_N^{(1)}(t, z_1) = \alpha \big(C_{1,2} f_N^{(2)} \big)(t, z_1)$$

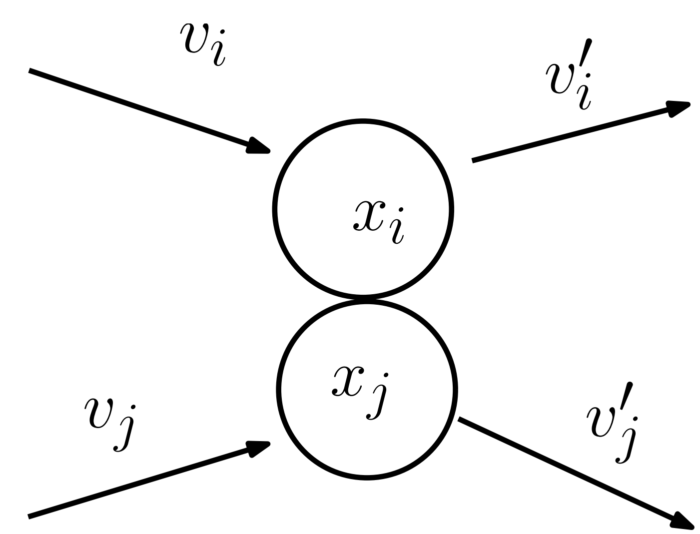
Collision operator

$$\begin{aligned}\big(C_{1,2} f_N^{(2)}\big)(z_1) :=& \int_{\mathbf{S}^{d-1}\times\mathbb{R}^d} f_N^{(2)}(x_1, v_1', x_1+\varepsilon\nu, v_2') \Big((v_2 - v_1)\cdot\nu \Big)_+ d\nu dv_2 \\ &- \int_{\mathbf{S}^{d-1}\times\mathbb{R}^d} f_N^{(2)}(x_1, v_1, x_1+\varepsilon\nu, v_2) \Big((v_2 - v_1)\cdot\nu \Big)_- d\nu dv_2\end{aligned}$$



BBGKY hierarchy for the marginals

Evolution of the first marginal



$$(\partial_t + v_1 \cdot \nabla_{x_1}) f_N^{(1)}(t, z_1) = \alpha \big(C_{1,2} f_N^{(2)}\big)(t, z_1)$$

Collision operator

$$\big(C_{1,2} f_N^{(2)}\big)(z_1) := \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(2)}(x_1, v_1', x_1 + \varepsilon\nu, v_2') \Big((v_2 - v_1)\cdot \nu\Big)_+ d\nu dv_2$$
$$- \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(2)}(x_1, v_1, x_1 + \varepsilon\nu, v_2) \Big((v_2 - v_1)\cdot \nu\Big)_- d\nu dv_2$$

Hope : Propagation of chaos

$$f_N^{(2)}(x_1, v_1, x_1 + \varepsilon\nu, v_2) \simeq f_N^{(1)}(x_1, v_1) f_N^{(1)}(x_1 + \varepsilon\nu, v_2)$$

Consequence: Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = \iint \left[f(v') f(v_1') - f(v) f(v_1) \right] \big((v - v_1)\cdot \nu\big)_+ dv_1 d\nu$$

BBGKY hierarchy for the marginals

For $s < N$ and on $\mathcal{D}_\varepsilon^s = \left\{ Z_s = (x_i, v_i)_{i \leq s} \,\middle|\, \quad i \neq j, \quad |x_i - x_j| > \varepsilon \right\}$

$$(\partial_t + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i}) f_N^{(s)}(t, Z_s) = \alpha \big(C_{s,s+1} f_N^{(s+1)} \big)(t, Z_s)$$

where the collision term is defined by

$$\begin{aligned}
&\big(C_{s,s+1} f_N^{(s+1)} \big)(Z_s) \\
&:= \frac{(N-s)\varepsilon^{d-1}}{\alpha} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(s+1)}(\dots, x_i, v_i', \dots, x_i + \varepsilon\nu, v_{s+1}') \Big((v_{s+1} - v_i) \cdot \nu \Big)_+ d\nu dv_{s+1} \\
&- \frac{(N-s)\varepsilon^{d-1}}{\alpha} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(s+1)}(\dots, x_i, v_i, \dots, x_i + \varepsilon\nu, v_{s+1}) \Big((v_{s+1} - v_i) \cdot \nu \Big)_- d\nu dv_{s+1}
\end{aligned}$$

where $\mathbf{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$.

Duhamel formula

Denote by $\mathbf{S}_s$ the semi-group associated to free transport in $\mathcal{D}^s_\varepsilon$

Duhamel Formula

$$f_N^{(1)}(t) = \mathbf{S}_1(t) f_N^{(1)}(0) + \alpha \int_0^t \mathbf{S}_1(t-t_1) C_{1,2} f_N^{(2)}(t_1)\, dt_1 \,,$$

Iterated Duhamel formula

$$f_N^{(1)}(t) = \sum_{n=0}^{N-1} \alpha^n \, Q_{1,1+n}(t) f_N^{(1+n)}(0)$$

Idea : Use the initial randomness

with

$$Q_{s,s+n}(t) := \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \, dt_n \dots dt_1 \, \mathbf{S}_s(t-t_1) C_{s,s+1}$$
$$\mathbf{S}_{s+1}(t_1-t_2) C_{s+1,s+2} \dots \mathbf{S}_{s+n}(t_n)$$

Duhamel formula

$$f_N^{(1)}(t) = \sum_{n=0}^{N-1} \alpha^n \, Q_{1,1+n}(t) f_N^{(1+n)}(0)$$

with

$$Q_{s,s+n}(t) := \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} dt_n \dots dt_1 \, \mathbf{S}_s(t-t_1) C_{s,s+1} \mathbf{S}_{s+1}(t_1 - t_2) C_{s+1,s+2} \dots \mathbf{S}_{s+n}(t_n)$$

Duhamel formula

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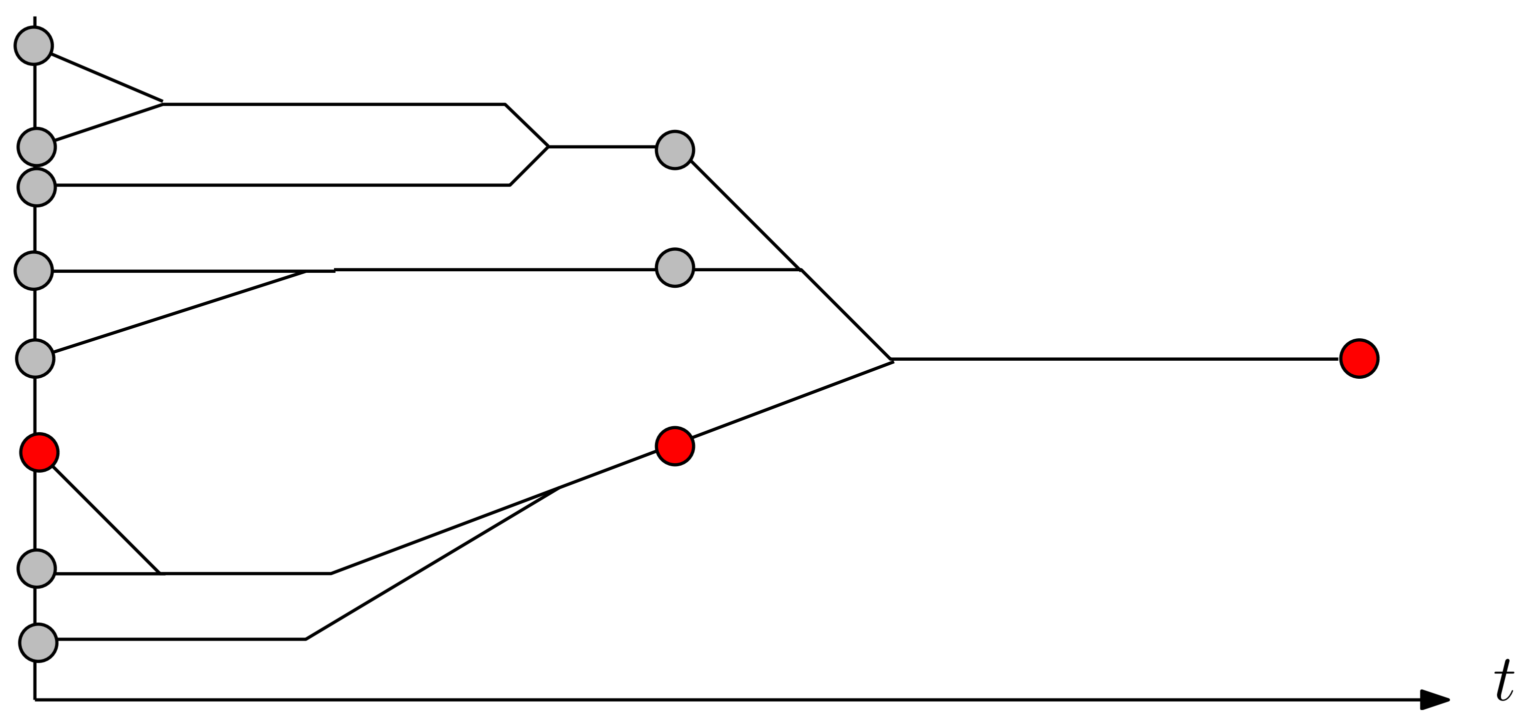
with

$$Q_{s,s+n}(t) := \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} dt_n \dots dt_1 \, \mathbf{S}_s(t-t_1) C_{s,s+1} \mathbf{S}_{s+1}(t_1-t_2) C_{s+1,s+2} \dots \mathbf{S}_{s+n}(t_n)$$

Interpretation as a collision tree

- Transport operator
- Addition of a particle to the tree after each collision



Issue : convergence of the series when N diverges

$$f_N^{(1)}(t) = \sum_{n=0}^{N-1} \alpha^n Q_{1,1+n}(t) f_N^{(1+n)}(0)$$

Continuity estimates for the collision operators

Weighted norms

$\|f_k\|_{\varepsilon,k,\beta} := \sup_{Z_k \in \mathcal{D}_\varepsilon^k} \left| f_k(Z_k) \;\exp\left(\frac{\beta}{2} \sum_{i=1}^k |v_i|^2\right)\right| < \infty$

Collision operators estimates

$$\left\| Q_{s,s+n}(t) f_{s+n} \right\|_{\varepsilon,s,\beta/2} \leq e^{s-1} (C_d(\beta) t)^n \|f_{s+n}\|_{\varepsilon,s+n,\beta}$$

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$$f_N^{(1)}(t) = \sum_{n=0}^{N-1} \alpha^n Q_{1,1+n}(t) f_N^{(1+n)}(0)$$

Series is only controlled for short times t and small α

Continuity estimates for the collision operators

Weighted norms

$$\|f_k\|_{\varepsilon,k,\beta} := \sup_{Z_k \in \mathcal{D}_\varepsilon^k} \left| f_k(Z_k) \, \exp\left(\frac{\beta}{2} \sum_{i=1}^k |v_i|^2\right) \right| < \infty$$

Collision operators estimates

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Continuity estimates for the collision operators

Weighted norms

$$\|f_k\|_{\varepsilon,k,\beta} := \sup_{Z_k \in \mathcal{D}^k_\varepsilon} \left| f_k(Z_k)\ \exp\left(\frac{\beta}{2}\sum_{i=1}^k |v_i|^2\right)\right| < \infty$$

Collision operators estimates

$$\left\| Q_{s,s+n}(t) f_{s+n} \right\|_{\varepsilon,s,\beta/2} \leq e^{s-1} (C_d(\beta) t)^n \|f_{s+n}\|_{\varepsilon,s+n,\beta}$$

$\mathbb{L}^\infty$ **bound**

Small perturbation of equilibrium

Initial distribution :

$$f_N^0(Z_N) = M_{N,\beta}(Z_N)\,\rho^0\big(x_1\big), \qquad \int_{\mathbb{T}^d_\lambda} dx_1 \rho^0\big(x_1\big) = 1$$

Uniform bound: $\rho^0\big(x_1\big) \leq \mu$

The measure $M_{N,\beta}(Z_N)$ is stationary thus the maximum principle implies bounds **uniform in time**

For any $s \geq 1$

$$\sup_{t\geq 0} f_N^{(s)}(t, Z_s) \leq \mu M_{N,\beta}^{(s)}(Z_s) \leq \mu\,\big(1 - \varepsilon c_d\big)^{-s}\ \ M_\beta^{\otimes s}(V_s)$$

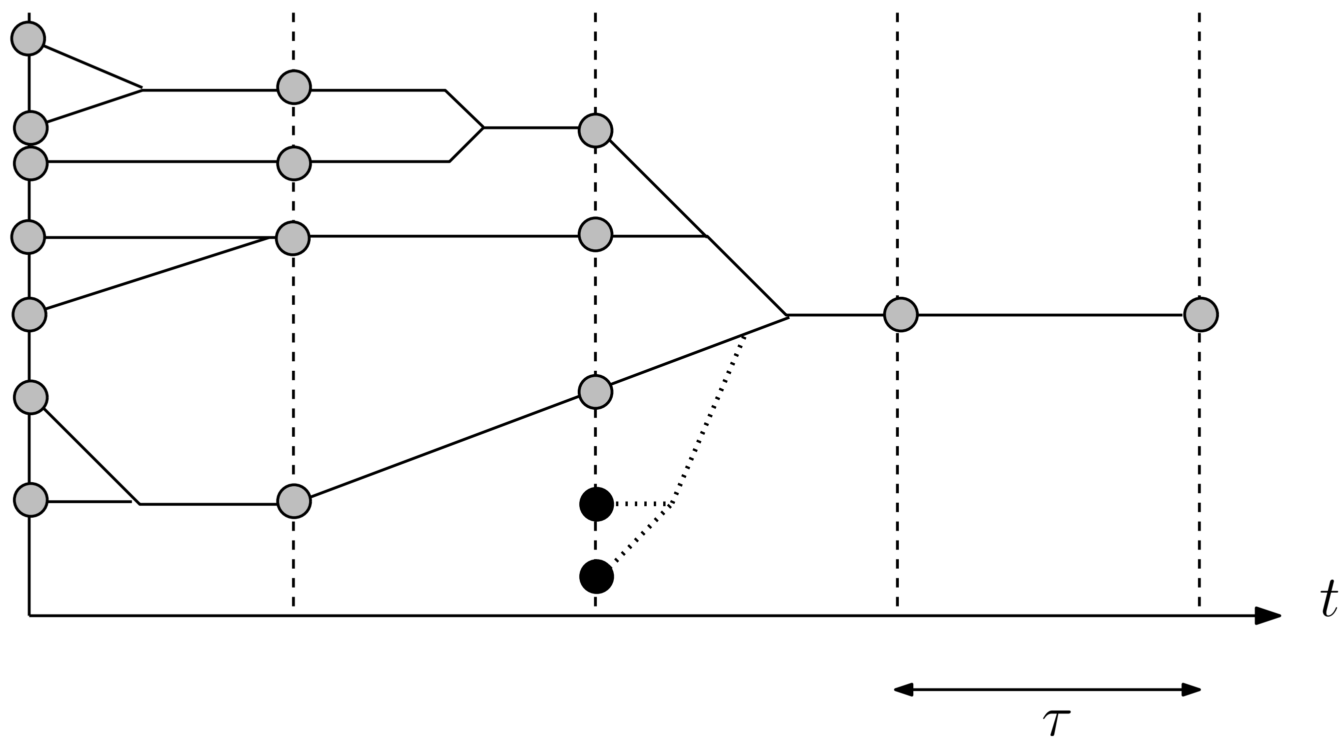
In this way the cancellations in the collision operator are recovered.

Pruning procedure

Decompose : $[0,t] = \bigcup_{k=1}^{K} [(k-1)\tau, k\tau]$ for some $\tau > 0$

Good collision trees.

Less than $n_k = 2^k$ collisions during $[(K-k)\tau, (K-k+1)\tau]$



In each time interval $[(K-k)\tau, (K-k+1)\tau]$

$$\left\| Q_{s,s+n}(\tau) f_{s+n} \right\|_{\varepsilon,s,\beta/2} \leq e^{s-1} \left(C_d(\beta)\tau \right)^n \| f_{s+n} \|_{\varepsilon,s+n,\beta}$$

Pruning procedure

Truncated iterated Duhamel formula:

$$f_N^{(1)}(t) = \sum_{j_1=0}^{2} \dots \sum_{j_K=0}^{2^K} \alpha^{J_k - 1} Q_{1,J_1}(\tau) Q_{J_1,J_2}(\tau) \dots Q_{J_{K-1},J_K}(\tau)\, f_N^{0(J_K)} + R_N^K(t)$$

with $\quad J_\ell = 1 + j_1 + \cdots + j_\ell$

- The main contribution is given by the good collision trees with $j_k \leq 2^k$ during the time interval $[(K-k)\tau, (K-k+1)\tau]$
- The contribution of the large trees $R_N^K(t)$ is controlled

$$\|R_N^K(t)\|_{\mathbb{L}^\infty} \leq \mu \frac{t^2}{K}$$

➪ If t is large, then K has to be very large and τ very small.

Derivation of the linear Boltzmann equation

Step 2. Comparison with the Boltzmann hierarchy

Boltzmann hierarchy

For $s \geq 1$ and $Z_s \in \mathbf{T}^{ds} \times \mathbb{R}^{ds}$

$$(\partial_t + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i}) g^{(s)}(t, Z_s) = \alpha \big(C^0_{s,s+1} \big) g^{(s+1)}(t, Z_s)$$

where the collision term is defined by

$$\begin{aligned}
&\big(C^0_{s,s+1} g^{(s+1)}\big)(Z_s) \\
&:= \cancel{(N-s)\varepsilon^{d-1}/\alpha} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} g^{(s+1)}(\dots, x_i, v_i^*, \dots, x_i \cancel{+ \varepsilon\nu}, v_{s+1}^*) \Big((v_{s+1} - v_i) \cdot \nu \Big)_+ d\nu dv_{s+1} \\
&\quad - \cancel{(N-s)\varepsilon^{d-1}/\alpha} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} g^{(s+1)}(\dots, x_i, v_i, \dots, x_i \cancel{+ \varepsilon\nu}, v_{s+1}) \Big((v_{s+1} - v_i) \cdot \nu \Big)_- d\nu dv_{s+1}
\end{aligned}$$

This is the **limit** hierarchy when $\varepsilon \to 0$ and $N \to \infty$.

Boltzmann hierarchy

For $s \geq 1$ and $Z_s \in \mathbf{T}^{ds} \times \mathbb{R}^{ds}$

$$(\partial_t + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i}) g^{(s)}(t, Z_s) = \alpha \big(C^0_{s,s+1} \big) g^{(s+1)}(t, Z_s)$$

Iterated Duhamel formula

$$g^{(1)}(t) = \sum_{n=0}^{\infty} \alpha^n Q^0_{1,1+n}(t) g^{(1+n)}(0)$$

Explicit solution : $g^{(s)}(t) = g(t, z_1) \ \prod_{i=2}^{s} M_\beta(v_i)$

with $g(t, z_1) = \varphi_\alpha(t, z_1) M_\beta(v_1)$ solution of the **Linear Boltzman equation**

$$\partial_t \varphi_\alpha + v \cdot \nabla_x \varphi_\alpha = -\alpha L \varphi_\alpha$$

and $M_\beta(v) := \left(\frac{\beta}{2\pi} \right)^{\frac{d}{2}} \exp\left(-\frac{\beta}{2} |v|^2 \right)$

Comparing the BBGKY and Boltzmann hierarchies

As $N \to \infty$ in the scaling $N\varepsilon^{d-1} = \alpha$,

$$\Big| \Big(f_N^{0(s)} - g^{0(s)} \Big) \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon} \Big| \leq C^s \varepsilon \alpha \, \mu \, M_\beta^{\otimes s}$$

Limiting product measure

for the **initial distributions**

$$f_N^0(Z_N) = M_{N,\beta}(Z_N) \, \rho^0(x_1), \quad g^{0(s)}(Z_s) = \Big(\prod_{i=1}^{s} M_\beta(v_i) \Big) \rho^0(x_1)$$

and

$$M_{N,\beta}(Z_N) = \frac{1}{\mathcal{Z}_{N,\beta}} \exp\Big(- \frac{\beta}{2} \sum_{i=1}^{N} |v_i|^2 \Big) \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon}$$

Main Goal

$$\| f_N^{(1)} - g^{(1)} \|_{L^\infty([0,t] \times \mathbf{T}^d \times \mathbb{R}^d)} \to 0 \,, \quad \text{as } N \to \infty$$

Comparing the truncated hierarchies

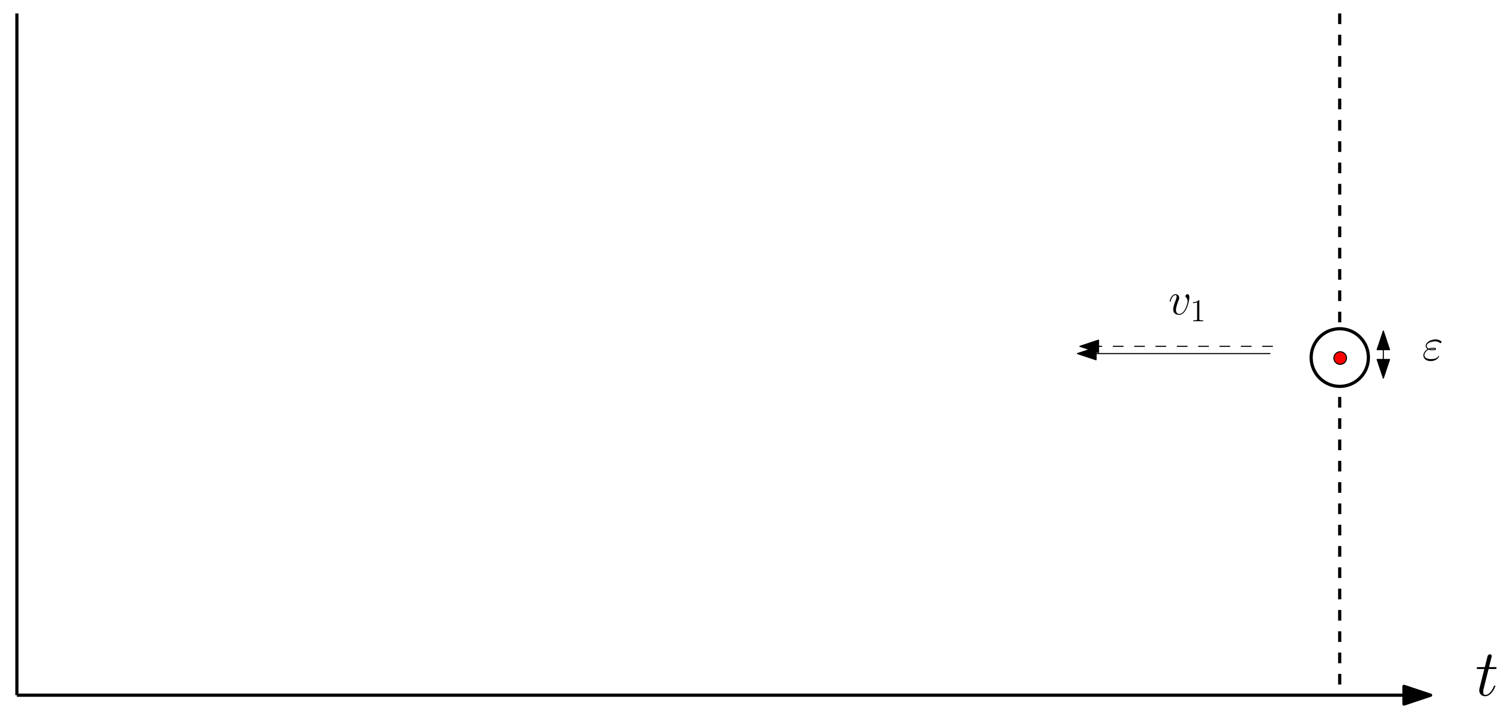
$$f_N^{(1,K)}(t) = \sum_{j_1=0}^{2} \dots \sum_{j_K=0}^{2^K} \alpha^{J_K} Q_{1,J_1}(\tau) Q_{J_1,J_2}(\tau) \dots Q_{J_{K-1},J_K}(\tau)\, f_N^{0(J_K)}$$

$$g^{(1,K)}(t) = \sum_{j_1=0}^{2} \dots \sum_{j_K=0}^{2^K} \alpha^{J_K} Q^0_{1,J_1}(\tau) Q^0_{J_1,J_2}(\tau) \dots Q^0_{J_{K-1},J_K}(\tau)\, g^{0(J_K)}$$

Geometric interpretation of the collisions operators:

Backward dynamics

Coupling the hierarchies



Comparing the truncated hierarchies

$$f_N^{(1,K)}(t) = \sum_{j_1=0}^{2} \dots \sum_{j_K=0}^{2^K} \alpha^{J_K} Q_{1,J_1}(\tau) Q_{J_1,J_2}(\tau) \dots Q_{J_{K-1},J_K}(\tau)\, f_N^{0(J_K)}$$

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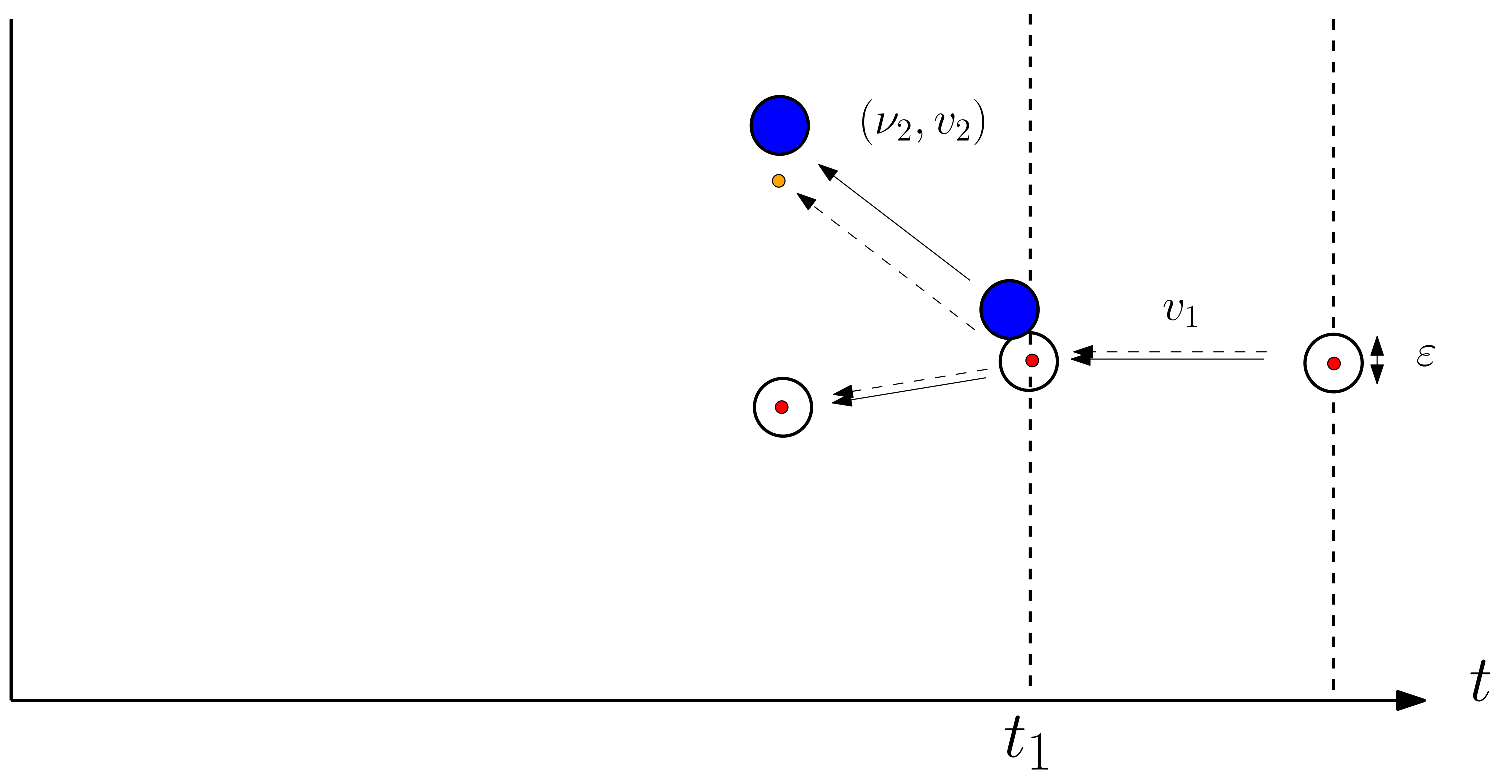
Geometric interpretation of the collisions operators:

Backward dynamics

Coupling the hierarchies



Comparing the truncated hierarchies

$$f_N^{(1,K)}(t) = \sum_{j_1=0}^{2} \dots \sum_{j_K=0}^{2^K} \alpha^{J_K} Q_{1,J_1}(\tau) Q_{J_1,J_2}(\tau) \dots Q_{J_{K-1},J_K}(\tau)\, f_N^{0(J_K)}$$

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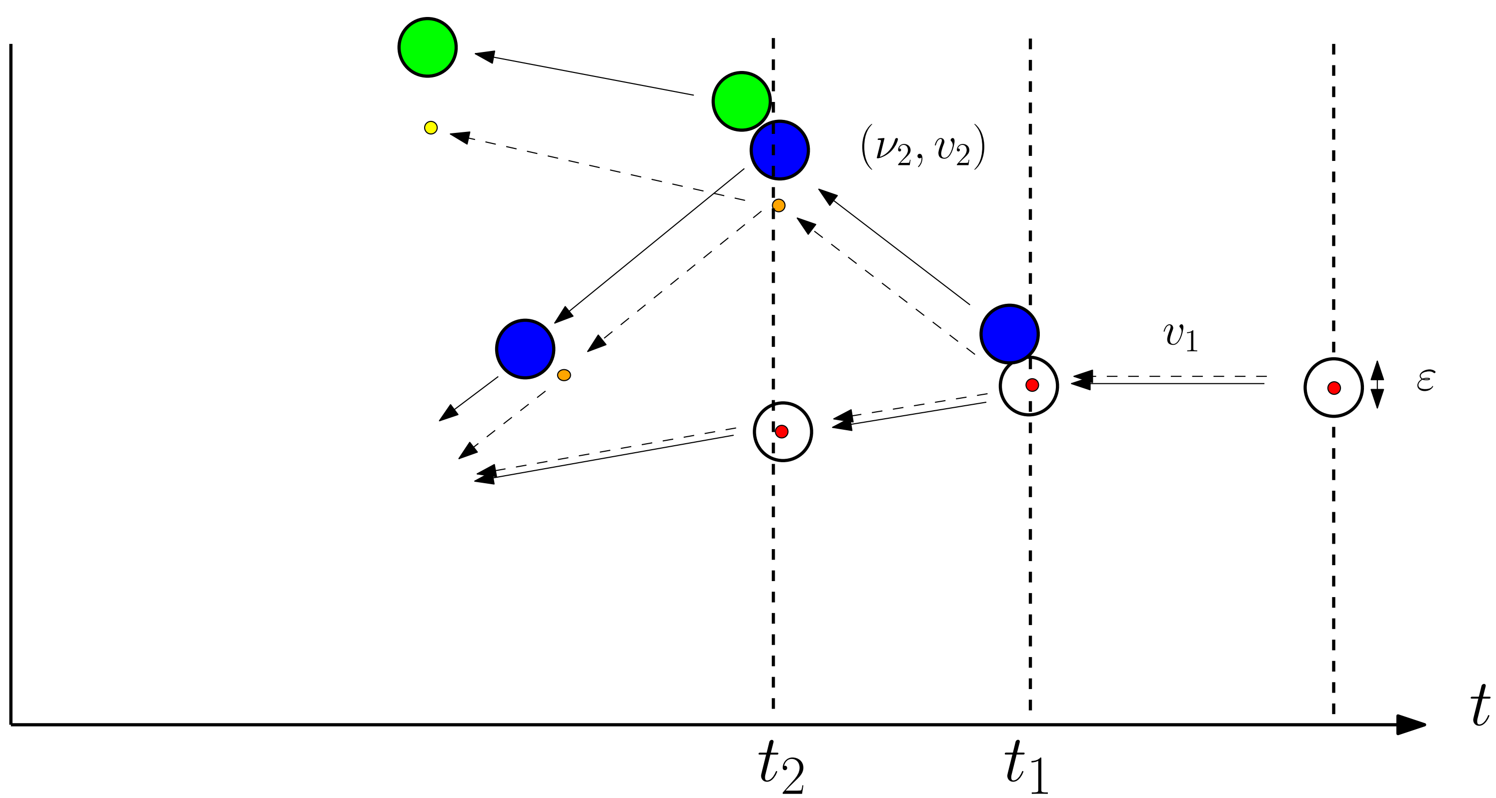
Geometric interpretation of the collisions operators:

Backward dynamics

Coupling the hierarchies



Comparing the truncated hierarchies

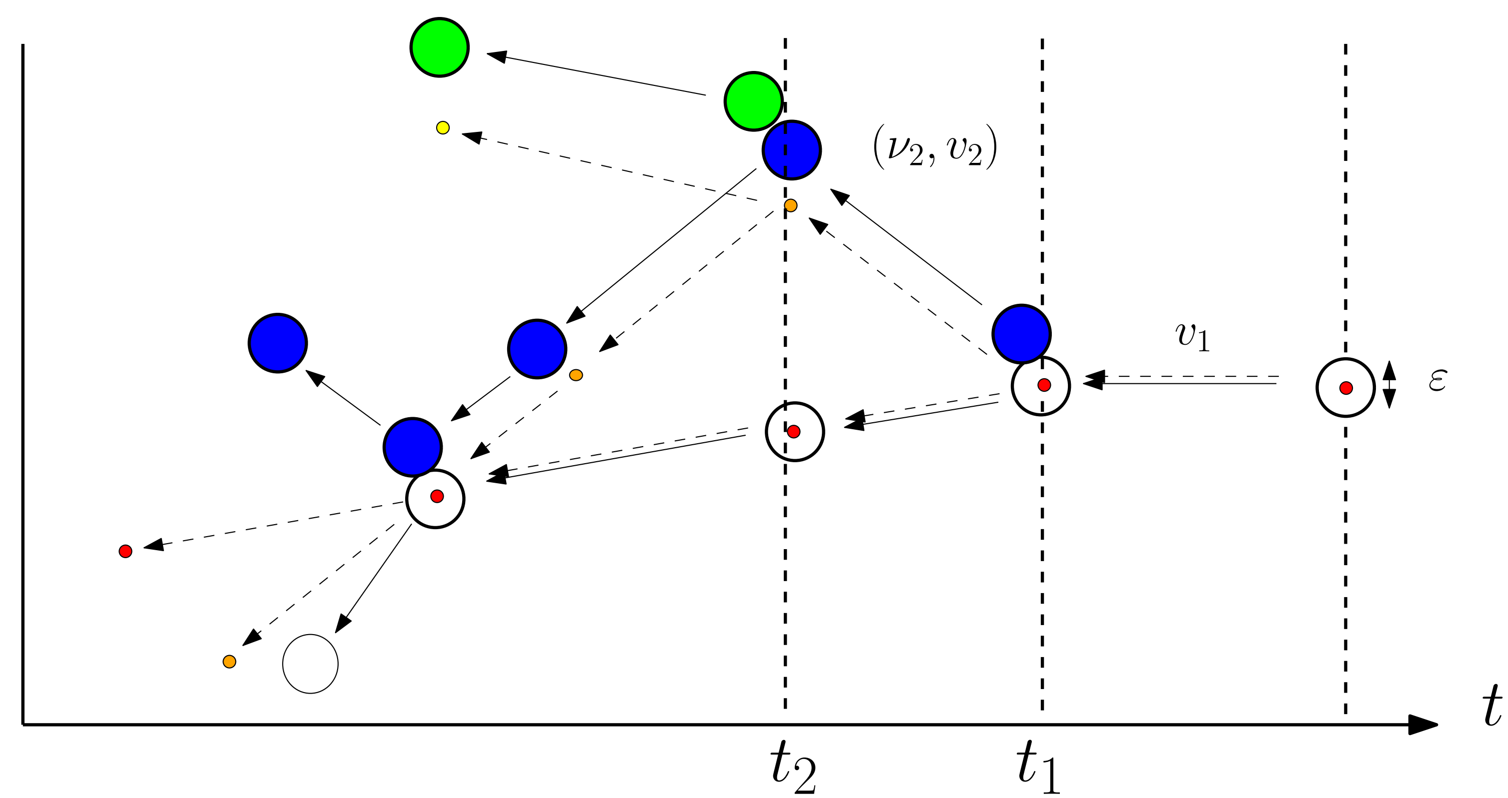
$$f_N^{(1,K)}(t) = \sum_{j_1=0}^{2} \dots \sum_{j_K=0}^{2^K} \alpha^{J_K} Q_{1,J_1}(\tau) Q_{J_1,J_2}(\tau) \dots Q_{J_{K-1},J_K}(\tau)\, f_N^{0(J_K)}$$

$$g^{(1,K)}(t) = \sum_{j_1=0}^{2} \dots \sum_{j_K=0}^{2^K} \alpha^{J_K} Q^0_{1,J_1}(\tau) Q^0_{J_1,J_2}(\tau) \dots Q^0_{J_{K-1},J_K}(\tau)\, g^{0(J_K)}$$

Geometric interpretation of the collisions operators:

Backward dynamics

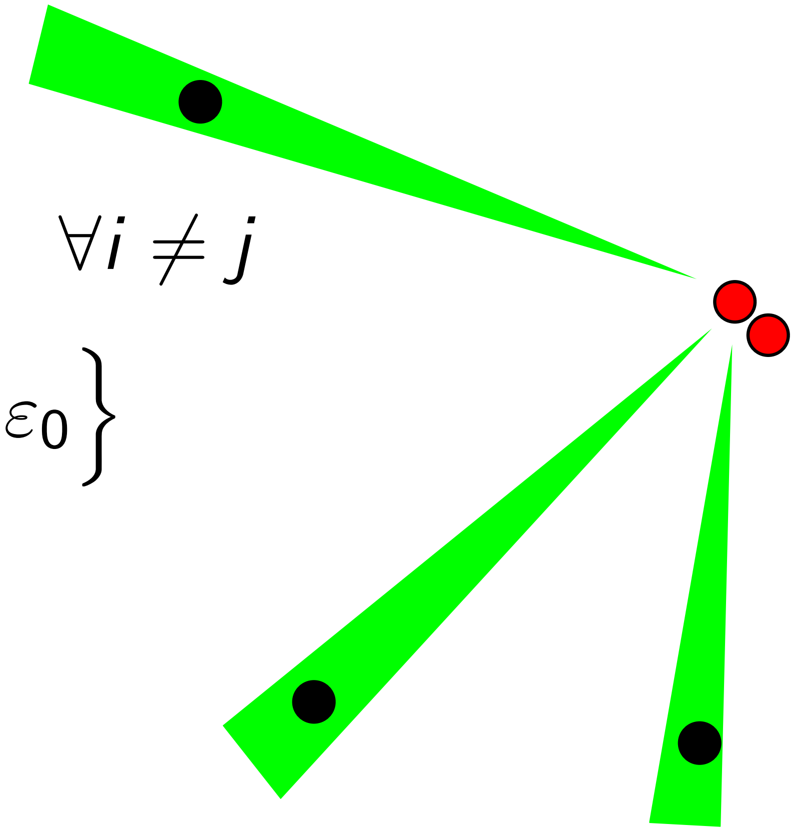
Coupling the hierarchies



Removing the collisions

BBGKY and Boltzmann trajectories can be coupled if there are no recollisions

- Truncating high velocities
- Truncating collisions in short time intervals
- Recursive construction of the good trajectories



$$\mathcal{G}_k(\varepsilon_0) = \Big\{ Z_k \in \mathbf{T}_\lambda^{dk} \times \mathbb{R}^{dk} \,/\, \forall s \in [0,t], \quad \forall i \neq j \quad d(x_i - sv_i, x_j - sv_j) \geq \varepsilon_0 \Big\}$$

Up to a small set of velocities, the system is stable by addition of the $k+1$ particle

Removing the collisions

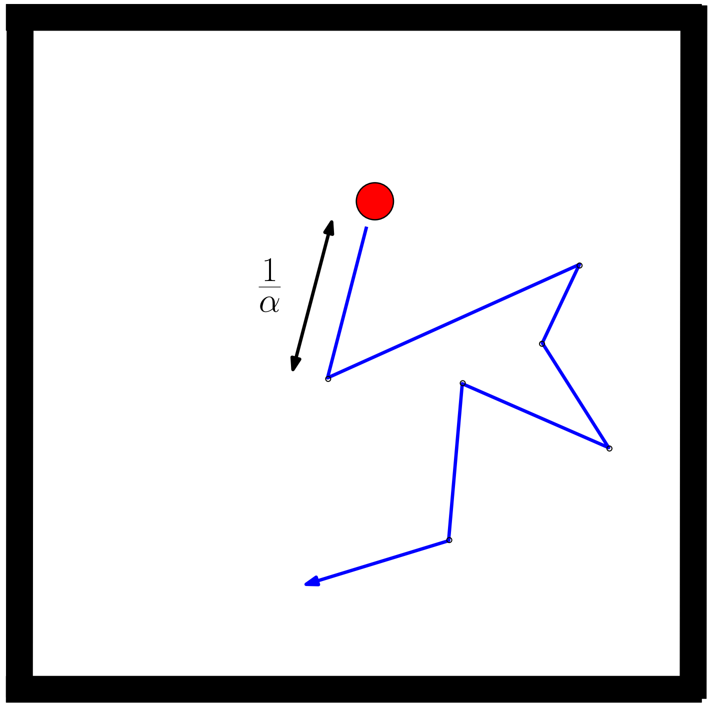
Choosing the velocities such that the particles in both hierarchies remain at distance less than $2^K\varepsilon$ and that there are no recollisions This boils down to remove a set of velocities with small probability.
Quantitative controls : [Gallagher, Saint-Raymond, Texier]

$$\textit{Error} \leq \Big(Ct\Big)^{\mathcal{N}_t}\varepsilon \quad \text{with} \quad \mathcal{N}_t \quad \text{particles in the tree at time } t$$

Estimates are valid up to times : $t_N = o\Big(\sqrt{\log\log(N)}\Big)$

$$\left\| f_N^{(1)} - g^{(1)} \right\|_{L^\infty([0,t_N]\times\mathbf{T}\times\mathbb{R}^d)} \leq C\mu\left(\frac{t_N^2}{\log\log N}\right)^2$$

Coupling both hierarchies



Position : $x_1^0(t) = \int_0^t v(u)du$

Markov process on the velocities $\{v(t)\}_{t\geq 0}$ with generator $\alpha\mathcal{L}$

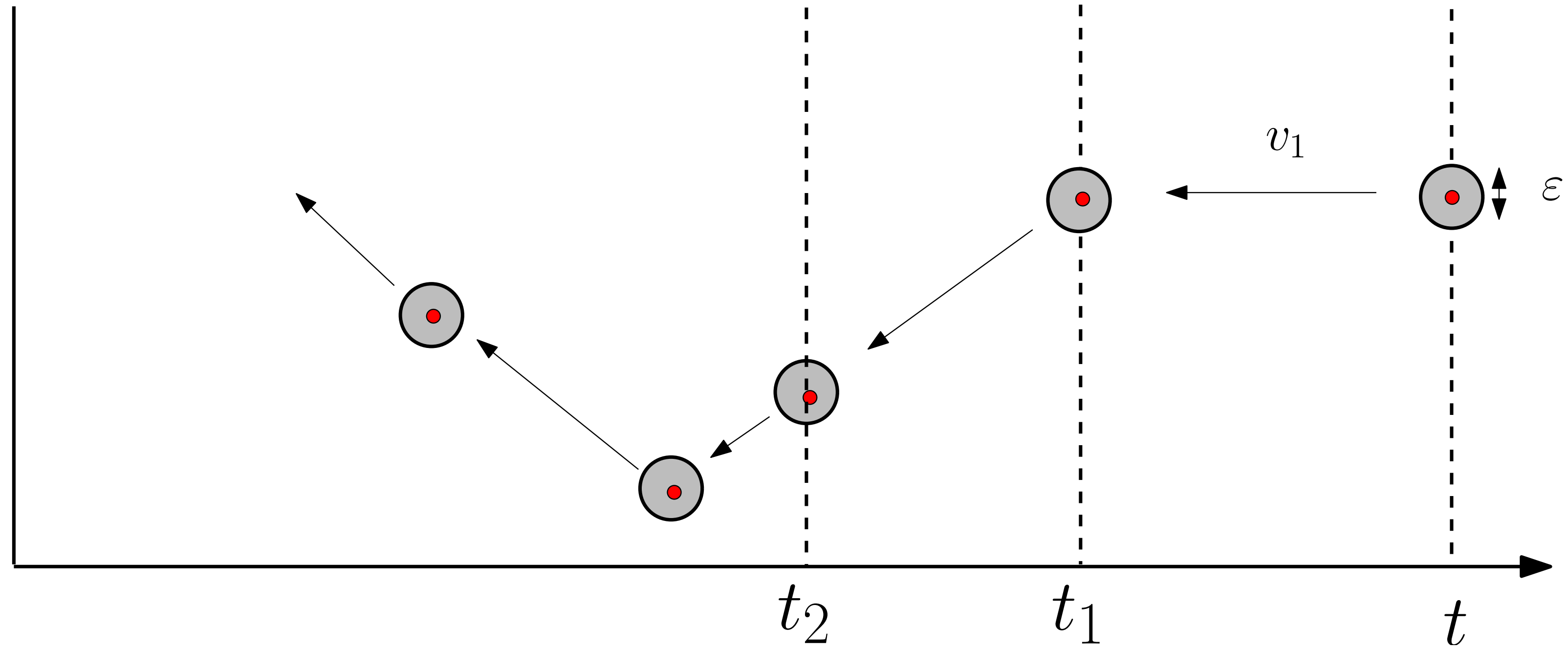
$$\begin{cases} \mathcal{L}g(v) := \iint M_\beta(v_1)[g(v') - g(v)] \big((v - v_1)\cdot\nu\big)_+ \, dv_1 d\nu, \\ v' = v + (\nu\cdot(v_1 - v))\,\nu \quad v_1' = v_1 - (\nu\cdot(v_1 - v))\,\nu \end{cases}$$

Central limit Theorem for additive functionals of Markov chains ($\mathcal{L}$ has a spectral gap)

$$\lim_{\alpha\to\infty} \mathbb{E}\Big(h\big(x_1^0(\alpha\,\tau)\big)\Big) = \mathbb{E}\Big(h\big(B(\tau)\big)\Big)$$

$$\lim_{\alpha \to \infty} \mathbb{E}\Big(h_1\big(x_1^0(\alpha\,\tau_1)\big) \dots h_\ell\big(x_1^0(\alpha\,\tau_\ell)\big) \Big) = \mathbb{E}\Big(h_1\big(B(\tau_1)\big) \dots h_\ell\big(B(\tau_\ell)\big) \Big)$$

Coupling the trajectories x_1 and x_1^0 to get estimates at different times



The diffusion coefficient κ_β is given by

$$\kappa_\beta = \frac{1}{d} \int_{\mathbb{R}^d} v \mathcal{L}^{-1} v \, M_\beta(v) \, dv$$

End of the first part !